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MIXED TYPE OF FUNCTIONAL EQUATION IN MATRIX NORMED SPACES

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Abstract. Using the fixed point method, we establish some stability results relating to the following mixed type additive-Quadratic functional equation

$$g(-u+2v)+2[g(3u-2v)+g(2u+v)-g(v)-g(v-u)] = 3[g(u+v)+g(u-v)+g(-u)]+4g(2u-v)$$

in matrix normed spaces.

1. Introduction and Preliminaries

A basic question in the theory of functional equations is as follows: When is it true that a function, which approximately satisfies a functional equation must be close to an act solution of the equation? If the problem accepts a solution, we say the equation is stable. The first stability problem concerning group homo morphisms was raised by Ulam [23] in 1940 and affirmatively solved by Hyers [9] . The result of Hyers was generalized by Aoki [1] for approximate additive mappings and by Rassias [17] for approximate linear mappings by allowing the difference Cauchy equation $\|f(x+y)-f(x)-f(y)\|$ to be controlled by $\epsilon(\|x\|^p+\|y\|^p)$. In 1994, a generalization of the Rassias theorem was obtained by Gavruta [7] who replaced $\epsilon(\|x\|^p+\|y\|^p)$ by a general control function $\psi(x,y)$. In addition, Rassias [18]-[21] generalized the Hyers stability result by introducing two weaker conditions controlled by a product of different powers of norms and a mixed product sum of powers of norms, respectively.

The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of *matricially normed spaces* [22] implies that quotients, mapping spaces and various tensor products of operator spaces may be treated as operator spaces. Owing this result, the theory of operator spaces is having a increasingly significant effect on operator algebra theory (see [5]).

The proof given in [22] appealed to the theory of ordered operator spaces [3]. Effros and Ruan [6] showed that one can give a purely metric proof of this important theorem by using a technique of Pisier [16] and Haagerup [8] (as modified in [4]).

Recently, J. R. Lee et al [13] researched the Ulam stability of Cauchy functional equation and quadratic functional equation in matrix normed spaces. This terminology may also be applied to

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the cases of other functional equations [10, 11, 12, 14, 24]. Here our purpose is to investigate to some stability result for the following equation

$$g(-u+2v)+2[g(3u-2v)+g(2u+v)-g(v)-g(v-u)] = 3[g(u+v)+g(u-v)+g(-u)]+4g(2u-v) \quad (1)$$

in matrix normed spaces by using the fixed point method.

2. General Solution of Mixed Type Functional Equation (1)

Lemma 1. *Let $\mathcal{G}$ and $\mathcal{H}$ be real vector spaces. If an odd mapping $g : \mathcal{G} \to \mathcal{H}$ satisfies (1), then g is additive.*

Proof. Suppose that g is an odd mapping, then the equation (1) is equivalent to

$$-g(u-2v) + 2[g(3u-2v) + g(2u+v) - g(v)] = 3[g(u+v) - g(u)] + g(u-v) + 4g(2u-v), \quad (2)$$

for all $u, v \in \mathcal{G}$. Put $u = u + v$ in (2), we obtain

$$-g(u-v) + 2[g(3u+v) + g(2u+3v) - g(v)] = 3[g(u+2v) - g(u+v)] + g(u) + 4g(2u+v) \quad (3)$$

for all $u, v \in \mathcal{G}$. Setting $(u, v) = (u + v, -v)$ in (3), we obtain

$$-g(u+2v) + 2[g(3u+2v) + g(2u-v) + g(v)] = 3[g(u-v) - g(u)] + g(u+v) + 4g(2u+v) \quad (4)$$

for all $u, v \in \mathcal{G}$. Subtracting (3) and (4), and then dividing the resulting equation by 2, one gets

$$-g(u+2v)+g(u-v)+g(2u+3v)-g(3u+2v)+g(3u+v)-g(2u-v) = -2g(u+v)+2g(u)+2g(v) \quad (5)$$

$\forall\ u, v \in \mathcal{G}$. Interchanging $u = v$ and $v = u$ in (5) and then adding the resulting equation to (5), one gets

$$-g(u+2v)-g(2u+v)+g(3u+v)+g(u+3v)-g(2u-v)+g(u-2v) = -4g(u+v)+4g(u)+4g(v) \quad (6)$$

Put $u = u - v$ in (6), we obtain

$$-g(u+v)-g(2u-v)+g(3u-2v)+g(u+2v)-g(2u-3v)+g(u-3v) = -4g(u)+4g(u-v)+4g(v) \quad (7)$$

for all $u, v \in \mathcal{G}$. Setting $(u, v) = (u, -v)$ in (7), we obtain

$$-g(u-v)-g(2u+v)+g(3u+2v)+g(u-2v)-g(2u+3v)+g(u+3v) = -4g(u)+4g(u+v)-4g(v) \quad (8)$$

for all $u, v \in \mathcal{G}$. Adding (7) and (8), we get

$$-g(u+2v)-g(2u+v)+g(3u+v)+g(u+3v)-g(2u-v)+g(u-2v) = -2g(u)+2g(u+v)-2g(v) \quad (9)$$

Subtracting (9) and (6), and then dividing the resulting equation by 6, one gets

$$g(u+v) = g(u) + g(v).$$

□

Lemma 2. *Let $\mathcal{G}$ and $\mathcal{H}$ be real vector spaces. If an even mapping $g : \mathcal{G} \to \mathcal{H}$ satisfies (1), then g is quadratic.*

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Proof. Suppose that g is an even mapping, then the equation (1) is equivalent to

$$g(u-2v)+2[g(3u-2v)+g(2u+v)-g(v)]=3[g(u+v)+g(u)]+5g(u-v)+4g(2u-v), \quad (10)$$

for all $u,v\in\mathcal{G}$. Put $u=u+v$ in (10) and $v=u+v$ in (10) and then comparing the two resulting equation, one gets

$$2g(u)+4g(u+2v)-2g(2u+3v)=-g(u+v)+5g(u-v)+3g(v)-g(2u+v)-2g(u-2v) \quad (11)$$

for all $u,v\in\mathcal{G}$. Interchanging $u=v$ and $v=u$ in (11), we obtain

$$2g(v)+4g(2u+v)-2g(3u+2v)=-g(u+v)+5g(u-v)+3g(u)-g(u+2v)-2g(2u-v) \quad (12)$$

for all $u,v\in\mathcal{G}$. Put $v=-v$ in(12), we get

$$2g(v)+4g(2u-v)-2g(3u-2v)=-g(u-v)+5g(u+v)+3g(u)-g(u-2v)-2g(2u+v) \quad (13)$$

Subtracting (13) and (10), and then dividing the resulting equation by 2, one gets

$$2g(v)+4g(2u-v)-2g(3u-2v)=-3g(u-v)+g(u+v) \quad (14)$$

for all $u,v\in\mathcal{G}$. Setting $(u,v)=(u+v,v)$ in (14), we get

$$g(u+2v)+2[g(3u+v)-g(v)]=3g(u)+4g(2u+v), \quad (15)$$

Setting $(u,v)=(u,v-u)$ in (15), we get

$$g(-u+2v)+2[g(2u+v)-g(v-u)]=3g(u)+4g(u+v) \quad (16)$$

for all $u,v\in\mathcal{G}$. Setting $(u,v)=(u,-v)$ in (16), we obtain

$$g(u+2v)+2[g(2u-v)-g(u+v)]=3g(u)+4g(u-v) \quad (17)$$

for all $u,v\in\mathcal{G}$. Replacing u by v and v by u in (16), we obtain that

$$g(2u-v)+2[g(u+2v)-g(u-v)]=3g(v)+4g(u+v) \quad (18)$$

for all $u,v\in\mathcal{G}$. Combining (17) and (18), and then divided by 3, one gets

$$g(u+2v)+g(2u-v)=g(u)+g(v)+2g(u+v)+2g(u-v) \quad (19)$$

for all $u,v\in\mathcal{G}$. Subtracting (17) from (18), and then adding the resulting equation (19), one gets

$$g(u+2v)+g(u)=2g(v)+2g(u+v) \quad (20)$$

for all $u,v\in\mathcal{G}$. Setting $(u,v)=(u-v,v)$ in (20), we arrive at $g(u+v)+g(u-v)=2g(u)+2g(v)$. This completes the proof. □

Throughout this paper, let $(X,\|.\|_n)$ be a matrix normed space, $(Y,\|.\|_n)$be a matrix Banach space and let n be a fixed positive integer.

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For a mapping $f : X \to Y$, define $\mathcal{M}f : X^2 \to Y$ and $\mathcal{M}g_n : M_n(X^2) \to M_n(Y)$ by,

$$\begin{aligned}\mathcal{M}g(p,q) = g(-p+2q) + 2[g(3p-2q) + g(2p+q) - g(q) - g(q-p)] \\ - 3[g(p+q) + g(p-q) + g(-p)] - 4g(2p-q),\end{aligned}$$

$$\begin{aligned}\mathcal{M}g_n([x_{ij}],[y_{ij}]) = g([-x_{ij}+2y_{ij}]) + 2[g([3x_{ij}-2y_{ij}]) + g([2x_{ij}+y_{ij}]) - g([y_{ij}]) - g([y_{ij}-x_{ij}])] \\ - 3[g([x_{ij}+y_{ij}]) + g([x_{ij}-y_{ij}]) + g([-x_{ij}])] - 4g([2x_{ij}-y_{ij}]),\end{aligned}$$

for all $p, q \in X$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

3. Generalized Hyers-Ulam Stability of (1): odd Case

Theorem 3. *Let $l = \pm 1$ be fixed and let $\psi : X^2 \to [0, \infty)$ be a function such that there exists a $\tau < 1$ with*

$$\psi(p,q) \le 2^l \tau \psi(\frac{p}{2^l}, \frac{q}{2^l}) \tag{21}$$

for all $a, b \in X$. Let $g : X \to Y$ be an odd mapping satisfying $g(0) = 0$ and

$$\|\mathcal{M}f_n([x_{ij}],[y_{ij}])\| \le \sum_{i,j=1}^{n} \psi(x_{ij}, y_{ij}) \quad \forall\ x = [x_{ij}], y = [y_{ij}] \in M_n(X). \tag{22}$$

Then there exists a unique additive mapping $\mathcal{A}_d : X \to Y$ such that

$$\|f_n([x_{ij}]) - \mathcal{A}_{dn}([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{\tau^{\left(\frac{1-l}{2}\right)}}{2\,|1-\tau|} \psi(0, x_{ij}) \quad \forall\ x = [x_{ij}] \in M_n(X). \tag{23}$$

Proof. Put $n = 1$ in (22) , we get

$$\|\mathcal{M}(p,q)\| \le \psi(p,q) \qquad \forall\ p, q \in X. \tag{24}$$

Letting $p = 0$ and $q = p$ in (24), we have

$$\|f(2p) - 2f(p)\| \le \frac{1}{2}\psi(0,p) \qquad \forall\ p \in X. \tag{25}$$

$$\text{So} \qquad \left\| f(p) - \frac{1}{2^l} f(2^l p) \right\| \le \frac{\tau^{\left(\frac{1-l}{2}\right)}}{2} \psi(0,p) \qquad \forall\ p \in X. \tag{26}$$

Let $\mathcal{T} = \{f : X \to Y\}$ and introduce the generalized metric ρ on $\mathcal{T}$ as follows:

$$\rho(f,g) = \inf \{\iota \in \mathbb{R}_+ : \|f(p) - g(p)\| \le \iota\psi(0,p), \forall p \in X\}.$$

It is easy to check that $(\mathcal{T}, \rho)$ is a complete generalized metric (see also [15]).
Define the mapping $\mathcal{E} : \mathcal{T} \to \mathcal{T}$ by $\mathcal{E}(p) = \frac{1}{2^l} f(2^l p)$ for all $f \in \mathcal{T}$ and $p \in X$. Let $f, g \in \mathcal{T}$ and ι be an arbitrary constant with $\rho(f,g) = \iota$. Then $\|f(p) - g(p)\| \le \iota\psi(0,p)$ for all $p \in X$. Hence

$\|\mathcal{E}f(p) - \mathcal{E}g(p)\| = \left\| \frac{1}{2^l} f(2^l p) - \frac{1}{2^l} g(2^l p) \right\| \le \tau\psi(0,p)$ for all $p \in X$.

This means that $\mathcal{E}$ is a contractive mapping with lipschitz constant $L = \tau < 1$.
It follows from (26) that $\rho(f, \mathcal{E}f) \le \dfrac{\tau^{\left(\frac{1-l}{2}\right)}}{|2|}$. By Theorem 2.2 in [2], there exists a mapping $\mathcal{A}_d : X \to Y$ which satisfying: 1. $\mathcal{A}_d$ is a fixed point of $\mathcal{E}$, i.e., $\mathcal{A}_d(2p) = 2\mathcal{A}_d(p)$

Journal of Pure and Applied Mathematics
Volume: 01, Issue: 01 July 2021, Page No. 1-8
ISSN: XXX-XXXX

2.$\rho(\mathcal{E}^k f, \mathcal{A}_d) \to 0$ as $k \to \infty$. This implies that $\lim_{k\to\infty} \frac{1}{2^{kl}} f(2^{kl}p) = \mathcal{A}_d(p) \quad \forall\ p \in X.$

3. $\rho(f, \mathcal{A}_d) \leq \dfrac{1}{1-\tau}\rho(f, \mathcal{E}f)$, this implies the inequality

$$\|f(p) - A_d(p)\| \leq \frac{\tau^{\frac{1-l}{2}}}{2\,|1-\tau|}\psi(0,p) \tag{27}$$

It follows from (21) and (24) that

$$\|\mathcal{M}\mathcal{A}_d(p,q)\| = \lim_{k\to\infty}\frac{1}{2^{lk}}\left\|\mathcal{M}f(2^{lk}p, 2^{lk}q)\right\| \leq \lim_{k\to\infty}\frac{1}{2^{lk}}\psi(2^{lk}p, 2^{lk}q) \leq \lim_{k\to\infty}\frac{2^{lk}\tau^k}{2^{lk}}\psi(p,q) = 0$$

Therefore

$-\mathcal{A}_d(p-2q)+2[\mathcal{A}_d(3p-2q)+\mathcal{A}_d(2p+q)-\mathcal{A}_d(p)]-3[\mathcal{A}_d(p+q)-\mathcal{A}_d(p)]+\mathcal{A}_d(p-q)-4\mathcal{A}_d(2p-q) = 0.$

Thus, the function $\mathcal{A}_d$ satisfies additive. Using Lemma 2.1 in [13] and (27), we get (23). Thus $\mathcal{A}_d : X \to Y$ is a unique additive mapping. □

Corollary 1. *Let $l = \pm 1$ be fixed and let r, ς be non-negative real numbers with $r \neq 1$. Let $g : X \to Y$ be a mapping such that*

$$\|\mathcal{M}f_n([x_{ij}],[y_{ij}])\|_n \leq \sum_{i,j=1}^{n} \varsigma(\|x_{ij}\|^r + \|y_{ij}\|^r) \quad \forall\ x = [x_{ij}], y = [y_{ij}] \in M_n(X). \tag{28}$$

Then there exists a unique additive mapping $\mathcal{A}_d : X \to Y$ such that

$$\|f_n([x_{ij}]) - \mathcal{A}_{dn}([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{\varsigma}{|2-2^r|}\|x_{ij}\|^r \qquad \forall\ x = [x_{ij}] \in M_n(X). \tag{29}$$

Proof. The proof follows from Theorem 3 by taking $\psi(p,q) = \varsigma(\|p\|^r + \|q\|^r)$ for all $p, q \in X$. Then we can choose $\tau = 2^{l(r-1)}$, and we get the desired result. □

4. Generalized Hyers-Ulam Stability of (1): Even Case

Theorem 4. *Let $l = \pm 1$ be fixed and let $\psi : X^2 \to [0,\infty)$ be a function such that there exists a $\tau < 1$ with*

$$\psi(p,q) \leq 4^l \tau \psi(\frac{p}{2^l}, \frac{q}{2^l}) \tag{30}$$

for all $p, q \in X$. Let $g : X \to Y$ be an even mapping satisfying $g(0) = 0$ and

$$\|\mathcal{M}g_n([x_{ij}],[y_{ij}])\| \leq \sum_{i,j=1}^{n} \psi(x_{ij}, y_{ij}) \quad \forall\ x = [x_{ij}], y = [y_{ij}] \in M_n(X). \tag{31}$$

Then there exists a unique quadratic mapping $\mathcal{Q}_d : X \to Y$ such that

$$\|g_n([x_{ij}]) - \mathcal{Q}_{dn}([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{\tau^{\left(\frac{1-l}{2}\right)}}{4\,|1-\tau|}\psi(0, x_{ij}) \quad \forall\ x = [x_{ij}] \in M_n(X). \tag{32}$$

Proof. Put $n = 1$. Then (31) is equivalent to

$$\|\mathcal{M}(p,q)\| \leq \psi(p,q) \qquad \forall\ p, q \in X. \tag{33}$$

Setting $p = 0$ and $q = p$ in (33), we get

$$\|g(2p) - 4g(p)\| \leq \frac{1}{2}\psi(0,p) \qquad \forall\ p \in X. \tag{34}$$

DBCY Publication Journal of Pure and Applied Mathematics
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$$\text{So} \qquad \left\| g(p) - \frac{1}{4^l} g(2^l p) \right\| \leq \frac{\tau^{\left(\frac{1-l}{2}\right)}}{4} \psi(0,p) \qquad \forall\ p \in X. \tag{35}$$

Let $(\mathcal{T}, \rho)$ be the generalized metric space defined in the proof of Theorem 3 . Now we consider the linear mapping $\mathcal{E} : \mathcal{T} \to \mathcal{T}$ defined by $\mathcal{E}(p) = \frac{1}{4^l} f(2^l p)$ for all $f \in \mathcal{T}$ and $p \in X$.

It follows from (35) that $\rho(f, \mathcal{E}f) \leq \dfrac{\tau^{\left(\frac{1-l}{2}\right)}}{|4|}$. So $\rho(f, Q_d) \leq \dfrac{\tau^{\frac{1-l}{2}}}{4\,|1-\tau|}$

The rest of the proof is similar to the proof of Theorem 3. □

Corollary 2. *Let $l = \pm 1$ be fixed and let r, ς be non-negative real numbers with $r \neq 2$. Let $g : X \to Y$ be a mapping such that*

$$\|\mathcal{M}g_n([x_{ij}], [y_{ij}])\|_n \leq \sum_{i,j=1}^{n} \varsigma(\|x_{ij}\|^r + \|y_{ij}\|^r) \quad \forall\ x = [x_{ij}], y = [y_{ij}] \in M_n(X). \tag{36}$$

Then there exists a unique quadratic mapping $\mathcal{Q}_d : X \to Y$ such that

$$\|g_n([x_{ij}]) - \mathcal{Q}_{dn}([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{\varsigma}{|4 - 2^r|} \|x_{ij}\|^r \qquad \forall\ x = [x_{ij}] \in M_n(X). \tag{37}$$

Proof. The proof follows from Theorem 4 by taking $\psi(p, q) = \varsigma(\|p\|^r + \|q\|^r)$ for all $p, q \in X$. Then we can choose $\tau = 2^{l(r-2)}$, and one can easily obtain the necessary result. □

5. J.M. Rassias Stability controlled by the mixed product-sum of powers of norms

The following corollary gives the Ulam J Rassias stability for the additive-quadratic functional equation (1). This stability involving the mixed product of sum of powers of norms.

Corollary 3. *Let $l = \pm 1$ be fixed and let r, ς be non-negative real numbers with $r = v + w \neq 1$. Let $g : X \to Y$ be a mapping such that*

$$\|\mathcal{M}g_n([x_{ij}], [y_{ij}])\|_n \leq \sum_{i,j=1}^{n} \varsigma(\|x_{ij}\|^v \cdot \|y_{ij}\|^w + \|x_{ij}\|^{v+w} + \|y_{ij}\|^{v+w}) \tag{38}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $\mathcal{A}_d : X \to Y$ such that (29).

Proof. The proof follows from Theorem 3 by taking $\psi(p, q) = \varsigma(\|p\|^v \cdot \|q\|^w + \|p\|^{v+w} + \|q\|^{v+w})$ for all $p, q \in X$. Then we can choose $\tau = 2^{l(r-1)}$, and we can obtain the required result. □

Corollary 4. *Let $l = \pm 1$ be fixed and let r, ς be non-negative real numbers with $r = v + w \neq 2$. Let $g : X \to Y$ be a mapping such that*

$$\|\mathcal{M}g_n([x_{ij}], [y_{ij}])\|_n \leq \sum_{i,j=1}^{n} \varsigma(\|x_{ij}\|^v \cdot \|y_{ij}\|^w + \|x_{ij}\|^{v+w} + \|y_{ij}\|^{v+w}) \tag{39}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quadratic mapping $\mathcal{Q}_d : X \to Y$ such that (37).

Proof. The proof follows from Theorem 4 by taking $\psi(p, q) = \varsigma(\|p\|^v \cdot \|q\|^w + \|p\|^{v+w} + \|q\|^{v+w})$ for all $p, q \in X$. □

6. Conclusion

Here, we found a general solution of a mixed type of Additive-quadratic functional equation (1) and established the generalized Hyers-Ulam stability and J. M. Rassias stability of the functional equation (1) in matrix normed spaces by using the fixed point method.

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