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STABILITY ANALYSIS OF A GENERALIZED EULER-LAGRANGE DUODEQUADRAGINTIC FUNCTIONAL EQUATION

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Abstract. In this paper, we introduce a new generalized version of the Euler-Lagrange duodequadragintic functional equation and investigate its Hyers-Ulam stability in fuzzy modular spaces (in short, FM-space) by using the fixed point approach.

1. Introduction and preliminaries

The famous field stability of functional equations was grown up from the question raised by Ulam [22] in 1940 and later it has a remarkable development by the various contributions of a number of authors importantly Hyers [4], Th. M. Rassias [17] and J. M. Rassias [11, 12, 13, 14]. In 1994, a generalized version of stabilities was obtained by Găvruta [2].

In the field of functional equations, different types of functional equations such as additive, quadratic, cubic, quartic, etc., have been given. After the developments in the field, mixed type functional equations evolved. A mixed type functional equation is nothing but an equation having more than one nature. The nature of the functional equation is determined by the solution which satisfies the equation.

Now we obtain a special kind of equation that shall possess the nature of any type of functional equation. We wish to name this type of equation as a multifarious functional equation. The equation can be expressed as

$$\begin{aligned} & f(ax_1+x_2)+f(ax_1-x_2)+f(x_1+ax_2)+f(x_1-ax_2) \\ & \quad -(a+a^2)[f(x_1+x_2)+f(x_1-x_2)] \\ & \quad = -2\left(a+a^2-a^{38}-1\right)f(x_1)-\left(1+(-1)^{38}\right)\left(a+a^2-a^{38}-1\right)f(x_2) \\ & \quad + \sum_{k\in 2\mathbb{N}}^{36} 76C_k\left(a^{38-k}+a^k-(a+a^2)\right)f\left(\sqrt[38]{x_1^{38-k}x_2^k}\right) \end{aligned} \tag{1}$$

for $a \neq 0, \pm 1$ and $k \in \mathbb{N}$.

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Importantly, this generalized functional equation is satisfied by $f(x) = x^{38}$. Therefore, it is called Euler-Lagrange radical duodequadragintic functional equation. The fuzzy modular space is developed from the concept of probabilistic modular space and the notion of fuzzy metric spaces in the sense of George and Veeramani [3].

In 2013, Shen and Chen[21] introduced the concept of FM-spaces. Later, Kumam [5, 6] and Wongkum *et al.* [23] applied the fixed point theory in FM-spaces and introduced some properties. In 2016, Wongkum and Kumam [24] investigated the Hyers-Ulam stability of sextic type functional equation in fuzzy modular space by considering the β-homogeneous and lower semi-continuous conditions.

In 2011, Ravi, Rassias and Narasimman [19] investigated the stability of a cubic functional equation in fuzzy normed space. Very recently, Rassias and Pasupathi [16] introduced Euler-Lagrange-Jensen general sextic functional equations and investigated its various Hyers-Ulam stabilities.

In 2019, Rassias, Dutta and Narasimman [15] investigated the stability of general A-quartic functional equations in non-Archimedean intuitionistic fuzzy normed spaces. Very recently, Murali, Divyakumari and Dutta [10] introduced an Euler-Lagrange radical functional equation with solution and stability. Also, various radical type functional equations were introduced by Murali and Divyakumari [7, 8, 9] in 2019.

Definition 1. (Fuzzy modular space [21]) Let η be a fuzzy set on $Z \times \mathbb{R}^+$, where Z is a complex or real vector space. Let Γ be a zero on Z and $\star$ be a continuous triangular norm. The ordered triple $(Z, \eta, \star)$ is said to be fuzzy modular space (in short, FM-space) and η is said to be fuzzy modular if it satisfies the following

(i) $0 < \eta(z, t)$;

(ii) $z = \Gamma$ if $\eta(z, t) = 1$;

(iii) $\eta(-z, t) = \eta(z, t)$;

(iv) $\eta(z, r) \star \eta(w, t) \leq \eta(\alpha z + \beta w, r + t)$, $\alpha, \beta \geq 0, \alpha + \beta = 1$;

(v) the mapping $\eta(z, \cdot) : (0, \infty) \to (0, 1]$ is continuous.

Definition 2. A fuzzy modular η is said to satisfy the Δ_2-condition if there exists $\kappa > 0$ such that $\eta(2z) = \kappa\eta(z)$ for all $z \in Z$.

Definition 3. An FM-space is said to be lower semi continuous if $\eta(z, t) \leq \lim_{p\to\infty} \inf \eta(z_p, t)$ for all $z \in Z, t > 0$, where $\{z_p\}$ is any sequence in Z and η-converges to z.

Example 4. Let η be a fuzzy set on $Z \times \mathbb{R}^+$, where Z is a complex or real vector space. Let $\star$ be a continuous triangular norm such that $\alpha \star \beta = \alpha \star_M \beta = min\{\alpha, \beta\}$. In that case, η is defined by

$$\eta(z, r) = \begin{cases} \frac{r}{r+\eta(z)} & , r > 0 \quad , z \in Z \\ 0, & otherwise. \end{cases}$$

This example holds even if we replace $\alpha \star \beta$ by $\alpha \star_P \beta$ and $\alpha \star_L \beta$.

In Sections 2, we obtain the general solution of the introduced duodequadragintic functional equation. We investigate the Hyers-Ulam stability of the introduced duodequadragintic functional equation in fuzzy modular space in Sections 3 by using fixed point aproach and the conclusion given in Section 4.

2. General solution of generalized Euler-Lagrange radical duodequadragintic functional equation

We obtain the general solution of (1) in the following form

$$\begin{aligned}
&f(ax_1+x_2)+f(ax_1-x_2)+f(x_1+ax_2)+f(x_1-ax_2)\\
&-(a+a^2)[f(x_1+x_2)+f(x_1-x_2)]\\
&=-2\left(a+a^2-a^{38}-1\right)f(x_1)-2\left(a+a^2-a^{38}-1\right)f(x_2)\\
&+\left(76C_2a^{36}+76C_2a^2-2(a+a^2)38C_2\right)f\left(\sqrt[38]{x_1^{36}x_2^2}\right)\\
&+\left(76C_4a^{34}+76C_4a^4-2(a+a^2)38C_4\right)f\left(\sqrt[38]{x_1^{34}x_2^4}\right)+\dots\\
&+\left(76C_{36}a^2+76C_{36}a^{36}-2(a+a^2)38C_{36}\right)f\left(\sqrt[38]{x_1^2x_2^{36}}\right)
\end{aligned} \tag{2}$$

for a fixed real a and $a\neq 0,\pm 1$.

Theorem 5. *Assume that X_1 and X_2 are real vector spaces. If $f: X_1\to X_2$ satisfies (2), then f is duodequadragintic and even.*

Proof. Assume that f satisfies (2). Replacing (x_1,x_2) by $(0,0)$ and $(x,0)$ in (2), respectively, we get $f(0)=0$ and

$$f(ax)=a^{38}f(x), \tag{3}$$

respectively. Hence f is duodequadragintic. Setting $x_1=0, x_2=x$ in (2) and using (3), we obtain $f(-x)=f(x)$ for all $x\in X_1$. Thus f is even. □

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3. Stability of generalized Euler-Lagrange radical duodequadragintic functional equation in FM-spaces

We prove the Hyers-Ulam stability of (1) in FM-spaces by using the fixed point technique. Assume, for $f : E \to (X, \eta)$,

$$\begin{aligned}
F(x_1, x_2) &= f(ax_1 + x_2) + f(ax_1 - x_2) + f(x_1 + ax_2) + f(x_1 - ax_2)\\
&- (a + a^2)[f(x_1 + x_2) + f(x_1 - x_2)]\\
&+ 2\left(a + a^2 - a^{38} - 1\right) f(x_1) + 2\left(a + a^2 - a^{38} - 1\right) f(x_2)\\
&- \left(76C_2a^{36} + 76C_2a^2 - 2(a + a^2)38C_2\right) f\left(\sqrt[38]{x_1^{36}x_2^2}\right)\\
&- \left(76C_4a^{34} + 76C_4a^4 - 2(a + a^2)38C_4\right) f\left(\sqrt[38]{x_1^{34}x_2^4}\right) - \dots\\
&- \left(76C_{36}a^2 + 76C_{36}a^{36} - 2(a + a^2)38C_{36}\right) f\left(\sqrt[38]{x_1^{2}x_2^{36}}\right),
\end{aligned}$$

for a fixed real a and $a \neq 0, \pm 1$.

Theorem 6. *Let E be a linear space, X be a real vector space and $(X, \eta, \star)$ be a η-complete β-homogeneous fuzzy modular space and $a \in \{-1, 1\}$ be fixed. Assume that a mapping $f : E \to (X, \eta, \star)$ satisfies*

$$\eta(F(x_1, x_2)) \geq \nu(x_1, x_2, t) \tag{4}$$

for all $x_1, x_2 \in E$ and a given function $\nu : E \times E \times (0, \infty) \to \Delta$, where Δ is the set of all non-decreasing function such that

$$\nu\left(a^b x_1, a^b x_2, a^{38\beta b} Nt\right) \geq \nu(x_1, x_2, t) \tag{5}$$

for all $x_1 \in E$ and

$$\lim_{m\to\infty} \nu\left(a^{bm} x_1, a^{bm} x_2, a^{38\beta bm} t\right) = 1 \tag{6}$$

for all $x_1, x_2 \in E$ and a constant $0 < N < \frac{1}{2^\beta}$. Then there exists a unique duodequadragintic mapping $M : E \to (X, \eta)$ satisfying (2) and

$$\eta\left(M(x_1) - f(x_2), \frac{t}{a^{38\beta} N^{\frac{b-1}{2}}(1 - 2^\beta N)}\right) \geq \nu(x_1, 0, t) \tag{7}$$

for all $x_1 \in E$.

Proof. Letting $x_2 = 0$ in (4), we obtain

$$\eta\left(2f(ax_1) - 2a^{38} f(x_1), t\right) \geq \nu(x_1, 0, t) \tag{8}$$

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and so

$$\eta\left(\frac{f(ax_1)}{\alpha^{38}}-f(x_1),t\right)\geq\nu\left(x_1,0,2^{\beta}a^{38\beta}t\right),\forall x_1\in E. \tag{9}$$

Replacing x_1 by $a^{-1}x_1$ in (9), we get

$$\eta\left(\frac{f(a^{-1}z)}{a^{-38}}-f(x_1),t\right)\geq\nu\left(x_1,0,2^{\beta}a^{38\beta}N^{-1}t\right). \tag{10}$$

Thus (9) and (10) imply

$$\eta\left(\frac{f(a^bx_1)}{a^{38b}}-f(x_1),t\right)\geq\nu\left(x_1,0,2^{\beta}a^{38\beta}N^{\frac{b-1}{2}}t\right),\forall x_1\in E. \tag{11}$$

Consider $P:=\{h:E\to(X,\eta)|h(0)=0\}$ and define ρ on P as follows:

$$\rho(h)=\inf\{l>0:\rho(h(x_1),lt)\geq\Psi(x_1,t),\quad\forall x_1\in E\}.$$

One can easily prove that ρ is a modular on P and indulges the Δ_2-condition with $2^{\beta}=\kappa$ and Fatou property. Additionally, P is ρ-complete (see [25]). Consider the mapping $R:P\to P$ as $RM(x_1):=\frac{M(a^bx_1)}{a^{38b}}$ for all $M\in P$.

Let $h,j\in P$ and $l>0$ be an arbitrary constant with $\rho(h-j)\leq l$. From the definition of ρ, we get

$$\Psi(x_1,t)\leq\eta(h(x_1)-j(x_1),lt)$$

and so

$$\eta(Rh(x_1)-Rj(x_1),Nlt)\geq\Psi(x_1,t),\,\forall x_1\in E.$$

Therefore, $\rho(Rh-Rj)\leq N\rho(h-j)$, $\forall h,j\in P$, that is, R is a ρ-strict contraction. Replacing x_1 by a^bx_1 in (11), we have

$$\eta\left(\frac{f(a^{2b}x_1)}{a^{38b}}-f(a^bx_1),t\right)\geq\Psi(a^bx_1,t) \tag{12}$$

for all $x_1\in E$ and therefore

$$\eta\left(a^{-2(38b)}f(a^{2b}x_1)-a^{-38b}f(a^bx_1),Nt\right)\geq\Psi(x_1,t),\,\forall x_1\in E. \tag{13}$$

Thus

$$\eta\left(\frac{f(a^{2b}x_1)}{a^{2(38b)}}-f(x_1),2^{\beta}(Nt+t)\right)\geq\Psi(x_1,t),\,\forall x_1\in E. \tag{14}$$

In (14), replacing x_1 by a^bx_1 and $2^{\beta}(Nt+t)$ by $a^{38\beta b}2^{\beta}(N^2t+Nt)$, we get

$$\eta\left(\frac{f(a^{3b}x_1)}{a^{2(38b)}}-f(a^bx_1),a^{38\beta b}2^{\beta}(N^2t+Nt)\right)\geq\Psi(x_1,t),\,\forall x_1\in E. \tag{15}$$

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Hence

$$\eta\left(\frac{f(a^{3b}x_1)}{a^{3(38b)}}-\frac{f(a^bx_1)}{a^{38b}},2^\beta(N^2t+Nt)\right)\geq\Psi(x_1,t),\ \forall x_1\in E, \tag{16}$$

which implies

$$\eta\left(\frac{f(a^{3b}x_1)}{a^{3(38b)}}-f(x_1),2^\beta\left(2^\beta(N^2t+Nt)+t\right)\right)\geq\Psi(x_1,t),\forall x_1\in E. \tag{17}$$

By (17), we obtain

$$\eta\left(\frac{f(a^{bm}x_1)}{a^{38(bm)}}-f(x_1),\left((2^\beta N)^{m-1}+2^\beta\sum_{i=1}^{m-1}(2^\beta N)^{i-1}\right)t\right)\geq\Psi(x_1,t) \tag{18}$$

for all $x_1\in E$ and a positive integer m. Hence we have

$$\rho(R^mf-f)\leq(2^\beta N)^{m-1}+2^\beta\sum_{i=1}^{m-1}(2^\beta N)^{i-1}\leq\frac{2^\beta}{1-2^\beta N}. \tag{19}$$

Now, one can easily prove that $\{R^m(f)\}$ is ρ-convergent to $M\in P$ (see[24]). Therefore, (19) becomes

$$\rho(M-f)\leq\frac{2^\beta}{1-2^\beta N}, \tag{20}$$

which means

$$\eta\left(M(x_1)-f(x_1),\frac{2^\beta}{1-2^\beta N}t\right)\geq\nu\left(x_1,0,2^\beta a^{38\beta}N^{\frac{b-1}{2}}t\right),\ \forall z\in E. \tag{21}$$

Hence we have

$$\eta\left(M(x_1)-f(x_1),\frac{t}{\alpha^{38\beta}N^{\frac{b-1}{2}}(1-2^\beta N)}\right)\geq\nu(x_1,0,t) \tag{22}$$

for all $x_1\in E$ and hence the inequality (7) holds. One can easily prove the uniqueness of M (see [24]). □

4. Conclusion

In this paper, we introduced a new generalized Euler-Lagrange radical duodequadragintic functional equation, which satisfies $f(x)=x^{38}$. This paper mainly dealt with general solution and the Hyers-Ulam stability of the generalized Euler-Lagrange radical duodequadragintic functional equation (1) in FM-spaces by using the fixed point approach.

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