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HYERS-ULAM-RASSIAS STABILITY OF A MIXED TYPE CUBIC-QUARTIC FUNCTIONAL EQUATION

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Abstract. In this paper, we investigate the Hyers-Ulam-Rassias Stability of a Mixed type Cubic-Quartic Functional Equation of the form

$$f(x+2y)+f(x-2y)=4\ f(x+y)+4\ f(x-y)-24\ f(y)-6\ f(x)+3\ f(2y)$$

in Banach spaces.

1. Introduction

The theory of stability is an important branch of the qualitative theory of functional equations. The concept of stability for a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. The first stability problem of functional equation was raised by S.M. Ulam [25] about seventy seven years ago. Since then, this question has attracted the attention of many researchers. Note that the affirmative solution to this question was given in the next year by D.H. Hyers [12] in 1941. In the year 1950, T. Aoki [1] generalized Hyers theorem for additive mappings. The result of Hyers was generalized independently by Th.M.Rassias [23] for linear mappings by considering an unbounded Cauchy difference. In 1994, a further generalization of Th.M. Rassias theorem was obtained by P.Gavruta [8].

After then, the stability problem of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem ([2, 3, 4, 13, 14, 16, 17, 19, 21, 22, 24, 29, 30]).

In 2002, Jun and Kim [15] introduced the following functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1}$$

and they established the general solution and the Hyers-Ulam-Rassias stability for the functional equation (1). The function $f(x)=x^3$ satisfies the functional equation (1), which is thus called a cubic functional equation and every solution of the cubic functional equation is said to be a cubic function.

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Park and Bae [18] introduced the following quartic functional equation

$$f(x+2y)+f(x-2y)=4\ f(x+y)+4\ f(x-y)+24\ f(y)-6\ f(x) \tag{2}$$

(see also [5, 6]). It is easy to show that the function $f(x)=x^4$ satisfies the functional equation (2), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.

Eshaghi Gordji and Bavand Savadkouhi [5] proved the Hyers-Ulam-Rassias stability of cubic and quartic functional equations in non-Archimedean normed space.

In this paper, we investigate the Hyers-Ulam-Rassias Stability of a Mixed type Cubic- Quartic Functional Equation of the form

$$f(x+2y)+f(x-2y)=4\ f(x+y)+4\ f(x-y)-24\ f(y)-6\ f(x)+3\ f(2y) \tag{3}$$

in Banach spaces. It is easy to show that the function $f(x)=x^3+x^4$ satisfies the functional equation (3), which is called a mixed type cubic-quartic functional equation. For more detailed definitions of mixed type functional equations, we can refer to [6, 7].

2. Main Results

Throughout this section, we assume that X is an additive group and Y is a Banach space. For a given $f : X \to Y$, we define the difference operator

$$Df(x,y)=f(x+2y)+f(x-2y)-4[f(x+y)-f(x-y)]+24\ f(y)+6\ f(x)-3\ f(2y)$$

for all $x, y \in X$. We consider the following function inequality:

$$\|Df(x,y)\| \leq \phi(x,y)$$

for an upper bound $\phi : X \times X \to [0,\infty)$.

Theorem 1. *Let $f : X \to Y$ be an odd mapping satisfying $f(0)=0$ for which there exists a function $\phi : X^2 \to [0,\infty)$ such that*

$$\lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{|2|^{3n}} = 0 \tag{4}$$

for all $x, y \in X$ and let for all $x \in X$, the limit

$$\phi_C(x) = \lim_{n\to\infty} \max\left\{\frac{\phi(0,2^j x)}{|2|^{3j}};\quad 0 \leq j \leq n\right\} \tag{5}$$

exists. Suppose that $f : X \to Y$ is an odd function satisfying

$$\|Df(x,y)\| \leq \phi(x,y) \tag{6}$$

for all $x, y \in X$. Then there exists a cubic function $C : X \to Y$ such that

$$\|C(x)-f(x)\| \leq \frac{1}{|3\ .\ 2^3|}\ \phi_C(x) \tag{7}$$

for all $x \in X$. Further more, if

$$\lim_{i\to\infty}\lim_{n\to\infty} \max\left\{\frac{\phi(0,2^j x)}{|2|^{3j}};\quad i \leq j \leq n+i\right\} = 0 \tag{8}$$

then C is the unique cubic function satisfies the inequality (7).

Proof. Putting $x = 0$ in (6), we have

$$\|3f(2y) - 24f(y)\| \leq \phi(0, y) \tag{9}$$

for all $y \in X$. If we replace y by x in (9) and divide it by 24 on both sides, we get

$$\left\| \frac{f(2x)}{2^3} - f(x) \right\| \leq \frac{1}{|3\ .\ 2^3|}\ \phi(0, x) \tag{10}$$

for all $x \in X$. Replacing x by $2^{n-1}x$ in (10), we get

$$\left\| \frac{1}{2^{3n}} f(2^n x) - \frac{1}{2^{3(n-1)}} f(2^{n-1}x) \right\| \leq \frac{1}{|3\ .\ 2^{3n}|}\ \phi(0, 2^{n-1}x) \tag{11}$$

for all $x \in X$. It follows from (11) and (4) that the sequence $\left\{ \frac{f(2^n x)}{2^{3n}} \right\}$ is a Cauchy sequence. Since X is complete, then we can conclude that $\left\{ \frac{f(2^n x)}{2^{3n}} \right\}$ is a convergent sequence.

Setting the limit,

$$C(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{3n}}$$

and using induction on n, we obtain that

$$\left\| \frac{f(2^n x)}{2^{3n}} - f(x) \right\| \leq \frac{1}{|3\ .\ 2^3|}\ \max\left\{ \frac{\phi(0, 2^i x)}{|2|^{3i}};\quad 0 \leq i \leq n \right\} \tag{12}$$

for $n \in \mathbb{N}$ and all $x \in X$. Applying $n \to \infty$ in (12), and using equation (5), we obtain that

$$\begin{aligned} \|C(x) - f(x)\| &\leq \frac{1}{|3\ .\ 2^3|}\ \lim_{n\to\infty} \max\left\{ \frac{\phi(0, 2^i x)}{|2|^{3i}};\quad 0 \leq i \leq n \right\} \\ &\leq \frac{1}{|3\ .\ 2^3|}\ \phi_C(x). \end{aligned}$$

Hence by using (4) and (6), we also have

$$\begin{aligned} \|Df(x, y)\| &= \lim_{n\to\infty}\ \frac{1}{|2^{3n}|}\ \|f(2^n x, 2^n y)\|\,| \\ &\leq \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{|2^{3n}|} = 0 \end{aligned}$$

for all $x, y \in X$. Hence the function $C : X \to Y$ satisfies the functional equation (3). Now, we have to prove that the uniqueness of the cubic function C. Let us choose C' is an another cubic function satisfies (7), then

$$\begin{aligned} \|C(x) - C'(x)\| &= \lim_{i\to\infty}\ |2|^{-3i}\ \left\|C(2^i x) - C'(2^i x)\right\| \\ &\leq \lim_{i\to\infty}\ |2|^{-3i}\ \max\left\{ \left\|C(2^i x) - f(2^i x)\right\|,\ \left\|f(2^i x) - C'(2^i x)\right\| \right\} \\ &\leq \frac{1}{|3\ .\ 2^3|}\ \lim_{i\to\infty} \lim_{n\to\infty} \max\left\{ \frac{\phi(0, 2^j x)}{|2|^{3j}};\quad i \leq j \leq n + i \right\} \end{aligned}$$

for all $x \in X$. If in addition,

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\left\{ \frac{\phi(0, 2^j x)}{|2|^{3j}};\quad i \leq j \leq n + i \right\} = 0,$$

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then one can have $C(x) = C'(x)$. This shows that $C(x)$ is unique. This completes the proof. □

Theorem 2. *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X \times X \to [0, \infty)$ such that*

$$\lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{|2|^{4n}} = 0 \tag{13}$$

for all $x, y \in X$ and let for each $x \in X$, the limit

$$\phi_Q(x) = \lim_{n\to\infty} \max\left\{ \frac{\phi(0, 2^j x)}{|2|^{4j}}; \quad 0 \le j \le n \right\} \tag{14}$$

exists. Suppose that $f : X \to Y$ is an odd function satisfying

$$\|Df(x, y)\| \le \phi(x, y) \tag{15}$$

for all $x, y \in X$. Then there exists a Quartic function $C : X \to Y$ such that

$$\|Q(x) - f(x)\| \le \frac{1}{|2^4|}\ \phi_Q(x) \tag{16}$$

for all $x \in X$. Further more, if

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\left\{ \frac{\phi(0, 2^j x)}{|2|^{4j}}; \quad i \le j \le n + i \right\} = 0 \tag{17}$$

then Q is the unique quartic function satisfying the inequality (16)*.*

Proof. Let us fixing $x = 0$ in (15), we get

$$\|f(2y) - 16f(y)\| \le \phi(0, y) \tag{18}$$

for all $y \in X$. If we replace y by x in (18) and divide it by 16 on both sides, then we get

$$\left\| \frac{f(2x)}{2^4} - f(x) \right\| \le \frac{1}{|2^4|}\ \phi(0, x) \tag{19}$$

for all $x \in X$. Replacing x by $2^{n-1}x$ in (19), we get

$$\left\| \frac{1}{2^{4n}} f(2^n x) - \frac{1}{2^{4(n-1)}} f(2^{n-1}x) \right\| \le \frac{1}{|2^{4n}|}\ \phi(0, 2^{n-1}x) \tag{20}$$

for all $x \in X$. It follows from (20) and (13) that the sequence $\left\{ \frac{f(2^n x)}{2^{3n}} \right\}$ is a Cauchy sequence. Since X is complete, then we can conclude that $\left\{ \frac{f(2^n x)}{2^{3n}} \right\}$ is a convergent sequence.
Now, setting the limit,

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{4n}}$$

by using induction on n, we obtain that

$$\left\| \frac{f(2^n x)}{2^{4n}} - f(x) \right\| \le \frac{1}{|2^4|}\ \max\left\{ \frac{\phi(0, 2^i x)}{|2|^{4i}}; \quad 0 \le i \le n \right\} \tag{21}$$

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for $n \in \mathbb{N}$ and all $x \in X$. Applying $n \to \infty$ in (21), and using equation (14), we obtain that

$$\begin{aligned}\left\|\frac{f(2^n x)}{2^{4n}} - f(x)\right\| &\leq \frac{1}{|2^4|} \lim_{n\to\infty} \max\left\{\frac{\phi(0,2^i x)}{|2|^{4i}}; \quad 0 \leq i \leq n\right\} \\ &\leq \frac{1}{|2^4|}\ \phi_Q(x)\end{aligned}$$

then we have

$$\begin{aligned}\|C(x) - f(x)\| &\leq \frac{1}{|2^4|} \lim_{n\to\infty} \max\left\{\frac{\phi(0,2^i x)}{|2|^{4i}}; \quad 0 \leq i \leq n\right\} \\ &\leq \frac{1}{|2^4|}\ \phi_Q(x).\end{aligned}$$

Hence by using (13) and (15), we also have

$$\begin{aligned}\|Df(x,y)\| &= \lim_{n\to\infty} \frac{1}{|2^{4n}|}\ \|f(2^n x, 2^n y)\|\,| \\ &\leq \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{|2^{4n}|} = 0\end{aligned}$$

for all $x, y \in X$. Hence the function $Q : X \to Y$ satisfies the functional equation (3). Now, we have to prove that the uniqueness of the quartic function Q. Let us choose Q' is an another quartic function satisfies (16), then

$$\begin{aligned}\|Q(x) - Q'(x)\| &= \lim_{i\to\infty} |2|^{-4i}\ \left\|Q(2^i x) - Q'(2^i x)\right\| \\ &\leq \lim_{i\to\infty} |2|^{-4i}\ \max\left\{\left\|Q(2^i x) - f(2^i x)\right\|,\ \left\|f(2^i x) - Q'(2^i x)\right\|\right\} \\ &\leq \frac{1}{|2^4|} \lim_{i\to\infty}\lim_{n\to\infty} \max\left\{\frac{\phi(0,2^j x)}{|2|^{3j}}; \quad i \leq j \leq n+i\right\}\end{aligned}$$

for all $x \in X$. If in addition,

$$\lim_{i\to\infty}\lim_{n\to\infty} \max\left\{\frac{\phi(0,2^j x)}{|2|^{3j}}; \quad i \leq j \leq n+i\right\} = 0,$$

then we have $Q(x) = Q'(x)$. This shows that $Q(x)$ is unique. This completes the proof. □

Theorem 3. *Let $\phi : X \times X \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{|2|^{3n}} = \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{|2|^{4n}} = 0 \tag{22}$$

for all $x, y \in X$ and let for each $x \in X$, the limit

$$\phi_C(x) = \lim_{n\to\infty} \max\left\{\frac{\phi(0,2^j x)}{|2|^{3j}}; \quad 0 \leq j \leq n\right\} \tag{23}$$

$$\phi_Q(x) = \lim_{n\to\infty} \max\left\{\frac{\phi(0,2^j x)}{|2|^{4j}}; \quad 0 \leq j \leq n\right\} \tag{24}$$

exists. Assume that $f : X \to Y$ is a function satisfying

$$\|Df(x,y)\| \leq \phi(x,y) \tag{25}$$

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for all $x, y \in X$. Then there exists a Cubic function $C : X \to Y$ and a Quartic function $C : X \to Y$ such that

$$\begin{aligned} &\|f(x) - C(x) - Q(x)\| \\ &\qquad \leq \frac{1}{|2^4|} \max\left\{\frac{1}{|3|} \max\{\phi_C(x), \phi_C(-x)\}, \frac{1}{|2|} \max\{\phi_Q(x), \phi_Q(-x)\}\right\} \end{aligned} \tag{26}$$

for all $x \in X$. Further more, if in addition,

$$\begin{aligned} &\lim_{i\to\infty}\lim_{n\to\infty} \max\left\{\frac{\phi(0, 2^j x)}{|2|^{3j}}; \quad i \leq j \leq n+i\right\} \\ &\qquad = \lim_{i\to\infty}\lim_{n\to\infty} \max\left\{\frac{\phi(0, 2^j x)}{|2|^{4j}}; \quad i \leq j \leq n+i\right\} = 0 \end{aligned} \tag{27}$$

then C is the unique cubic function and Q is the unique quartic function.

Proof. Assume that $f_o(x) = \frac{1}{2}[f(x) - f(-x)]$ for all $x \in X$. Then $f_o(0) = 0$ and $f_o(-x) = -f_o(x)$ with

$$\|Df_o(x, y)\| \leq \frac{1}{|2|} \max\{\phi(x, y), \phi(-x, -y)\} \tag{28}$$

for all $x \in X$. From Theorem 1 we have that there exists a unique cubic function $C : X \to Y$ satisfying

$$\|f_o(x) - C(x)\| \leq \frac{1}{|3 \,.\, 2^4|} \max\{\phi_C(x), \phi_C(-x)\} \tag{29}$$

for all $x \in X$. Now, let $f_e(x) = \frac{1}{2}[f(x) + f(-x)]$ for all $x \in X$. Then $f_e(0) = 0$ and $f_e(-x) = f_e(x)$ with

$$\|Df_e(x, y)\| \leq \frac{1}{|2|} \max\{\phi(x, y), \phi(-x, -y)\} \tag{30}$$

for all $x \in X$. From Theorem 2 we have that there exists a unique Quartic function $Q : X \to Y$ satisfying the inequality

$$\|f_e(x) - Q(x)\| \leq \frac{1}{|2^5|} \max\{\phi_Q(x), \phi_Q(-x)\}. \tag{31}$$

for all $x \in X$. Hence by using the inequalities (29) and (31), we reach at (26). This completes the proof. $\square$

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