DBCY Publication

Journal of Pure and Applied Mathematics
Volume: 01, Issue: 01 July 2021, Page No. 36-48
ISSN: XXX-XXXX

DELTA FRACTIONAL PROPORTIONAL DIFFERENCES ON $h\mathbb{Z}$ AND THE $h-$LAPLACE TRANSFORM

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Abstract. The study of delta fractional proportional differences on $h\mathbb{Z}$ as a bridge between fractional calculus in the literature and on a time scale $h\mathbb{Z}$. By use of time scale calculus notation, we formulate delta fractional sums and differences on fractional difference operator. The discrete $h-$Laplace transform and its convolution are used to study the newly introduced fractional operators. Numerical examples with graphs are verified and generated by MATLAB.

1. Introduction

The fractional difference operator which plays a pivotal role in studying numerous systems and has been widely applied in various areas of study [6, 11, 13, 14]. The above theory found its advent with the invention of the difference operator Δ_ℓ, which is an extension of Δ. It helps in drawing the higher partial sums on arithmetic, geometric progression and products of n-consecutive terms of arithmetic progression [15]. A better sense of direction in this field was given by Maria Susai Manuel et.al., in 2011 with the extension of $\underset{\alpha}{\Delta}$ to α-difference operator $\underset{\alpha(\ell)}{\Delta}$, where $u(k+\ell)-\alpha u(k) = \underset{\alpha(\ell)}{\Delta} u(k)$, $\ell > 0$. Further development in this line of study was further complemented by the works of Britto Antony Xavier et.al., [4, 5]. When $\ell = 1$, the operator Δ_ℓ becomes Δ.

2. The delta fractional sums and differences

Here, we present some basic definitions, notations and preliminaries.

Definition 1. Let $u(t),\ t \in [0, \infty)$, be a real or complex valued function and $h > 0$ be a fixed shift value. Then, the forward difference operator on $h\mathbb{Z}$ is defined as

$$\Delta_h u(t) = \frac{u(t+h) - u(t)}{h}, \tag{1}$$

and the backward difference operator on $h\mathbb{Z}$ is defined as

$$\nabla_h u(t) = \frac{u(t) - u(t-h)}{h}. \tag{2}$$

For $h = 1$, we get $\Delta u(t) = u(t+1) - u(t)$ and $\nabla u(t) = u(t) - u(t-1)$ respectively.

2010 *Mathematics Subject Classification.* : 65T50, 39A10, 43A50, 42A38, 42B10, 26A33.

Key words and phrases. : Time scale, fractional calculus Laplace transform and convolution.

Journal of Pure and Applied Mathematics
Volume: 01, Issue: 01 July 2021, Page No. 36-48
ISSN: XXX-XXXX

The forward jumping operator on the time scale $h\mathbb{Z}$ is $\sigma_h(t) = t+h$ and the backward jumping operator is $\rho_h(t) = t - h$. For $a, b \in R$ and $h > 0$, we use the notation $\mathbb{N}_{a,h} = \{a, a+h, a+2h, ...,\}$ and ${}_{b,h}\mathbb{N} = \{b, b-h, b-2h, ...\}$.

Definition 2. Let $u(t)$ and $v(t)$ are the two real valued functions defined on $(-\infty, \infty)$ and if $\Delta_h v(t) = u(t)$ then the finite inverse principle law is given by

$$v(t) - v(t - mh) = h \sum_{r=1}^{m} u(t - rh), m \in Z^+ \tag{3}$$

and infinite $h-$ difference sum is defined by

$$\Delta_h^{-1} u(t) = h \sum_{r=0}^{\infty} u(t + rh). \tag{4}$$

Definition 3. [7] For $h > 0$ and $\nu \in R$, the falling h-polynomial factorial function is defined by

$$t_h^{(\nu)} = h^\nu \frac{\Gamma(\frac{t}{h} + 1)}{\Gamma(\frac{t}{h} + 1 - \nu)}. \tag{5}$$

For $h = 1$, we get $t^{(\nu)} = \dfrac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$. The division by a pole yields zero.

Remark 4. Applying the Definition 1, we get the modified identities as follows:

$$(i)\ \Delta_h t_h^{(\mu)} = \mu t_h^{(\mu-1)}, \quad (ii)\ \Delta_h^{-1} t_h^{(\mu)} = \frac{t_h^{(\mu+1)}}{\mu + 1}. \tag{6}$$

Since the difference equation $\Delta_h v(t) = u(t)$ has two kinds of solution: one is closed form solution of the form $v(t) = \Delta_h^{-1} u(t)$ and another one is summation form solution which is of the form $v(t) = \sum\limits_{i=a/h}^{t/h-1} u(ih)$. In this work, we verify the findings are equal numerically in both kind of solutions. For getting the closed form solution for the product of two function, we need the following the discrete integration by parts.

Lemma 5. [9] *Let $h > 0$ and $u(t)$, $w(t)$ are real valued bounded functions. Then*

$$(i) \quad \Delta_h^{-1}(u(t)v(t)) = u(t)\Delta_h^{-1}v(t) - \Delta_h^{-1}(\Delta_h^{-1}v(\sigma_h(t))\Delta_h u(t)). \tag{7}$$

$$(ii) \quad \Delta_h^{-1}[v(\sigma_h(t))\Delta_h u(t)] = u(t)v(t) - u(t)\Delta_h v(t). \tag{8}$$

In view[12], the equations (7) and (8) can be written as the following integration by parts theorem.

Definition 6. (**Integration by parts**) Given two functions $u, v : \mathbb{N}_{a,h} \to R$ and $b, c \in \mathbb{N}_{a,h}, b < c$, then we have

$$(i) \quad \int_b^c u(t)\Delta_h v(t)\Delta_h t = u(t)v(t)\Big|_b^c - \int_b^c v(\sigma_h(t))\Delta_h u(t)\Delta_h t \tag{9}$$

$$(ii) \quad \int_b^c u(\sigma_h(t))\Delta_h v(t)\Delta_h t = u(t)v(t)\Big|_b^c - \int_b^c v(t)\Delta_h u(t)\Delta_h t \tag{10}$$

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Volume: 01, Issue: 01 July 2021, Page No. 36-48
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In case $h = 1$, we obtain the existing integration by parts theorem in [12].

Lemma 7. *Let $h > 0$, $s \in R$ and $hs \neq -1$. Then*

$$_a\Delta_h^{-1}\left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}+1} = \frac{h\left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}+1}}{\left(\frac{-hs}{1+hs}\right)}. \tag{11}$$

Proof. The proof follows by taking $u(t) = \left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}+1}$ in (1) and applying $_a\Delta_h^{-1}$ on both sides. ◻

Corollary 1. *Let $h > 0$, $s \in R$ and $hs \neq -1$. Then*

$$\frac{\left(\frac{1}{1+hs}\right)^{\frac{t-mh-a}{h}}}{s} - \frac{\left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}}}{s} = h\sum_{r=1}^{m}\left(\frac{1}{1+hs}\right)^{\frac{t-rh-a}{h}+1}. \tag{12}$$

Proof. The proof of (12) follows by applying (3) in (11). ◻

Example 8. For the particular values $a = 2, h = 3, s = 4, t = 8$ and $m = 3$, the corollary 1 is verified by MATLAB, numerically $LHS = RHS = 3.2485$.

Lemma 9. *Let $s \in \mathbb{T} = \mathbb{N}_{a,h}$, then for all t we have*

$$\Delta_{h;t}\frac{(t-s)_h^{(k+1)}}{(k+1)!} = \frac{(t-s)_h^{(k)}}{k!} \tag{13}$$

Proof. The proof follows by using definitions and direct calculations,
where $\Delta_{h;t}u(t,s) = \frac{u(t+h,s)-u(t,s)}{h}$. ◻

Lemma 10. *For the time scale $T = \mathbb{N}_{a,h}$, the delta Taylor polynomial is*

$$h_n(t,s) = \frac{(t-s)_h^{(n)}}{n!}, \quad n \in \mathbb{N}_0 \tag{14}$$

Definition 11. [3] (Delta discrete Mittag-Leffler) For $\lambda \in R, |\lambda| < 1$ and $\alpha, \beta, z \in \mathbb{C}$ with $Re(\alpha) > 0$, the delta discrete Mittag-Leffler function is

$$E_{(\alpha,\beta)}(\lambda, z) = \sum_{k=0}^{\infty}\lambda^k\frac{(z+(k-1)(\alpha-1))^{(k\alpha)}(z+k(\alpha-1))^{(\beta-1)}}{\Gamma(\alpha k+\beta)}. \tag{15}$$

For $\beta = 1$, we have

$$E_{(\alpha)}(\lambda, z) = \sum_{k=0}^{\infty}\lambda^k\frac{(z+(k-1)(\alpha-1))^{(k\alpha)}}{\Gamma(\alpha k+1)}. \tag{16}$$

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Definition 12. (Delta h-discrete Mittag-Leffler) For $\lambda \in R, |\lambda h^{\alpha}| < 1$ and $\alpha, \beta, z \in \mathbb{C}$ with $Re(\alpha) > 0$, the delta discrete Mittag-Leffler function is

$$ {}_hE_{(\alpha,\beta)}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{(z + (k-1)(\alpha-1))_h^{(k\alpha)} (z + k(\alpha-1))_h^{(\beta-1)}}{\Gamma(\alpha k + \beta)}. \tag{17} $$

For $\beta = 1$, we have

$$ {}_hE_{(\alpha)}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{(z + (k-1)(\alpha-1))_h^{(k\alpha)}}{\Gamma(\alpha k + 1)}. \tag{18} $$

Definition 13. [8] (Delta $h-$fractional sums) For a function $u : \mathbb{N}_{a,h} \to R$, the delta left $h-$fractional sum of order $\nu > 0$ is given by

$$ \begin{aligned} ({}_a\Delta_h^{-\nu} u)(t) &= \frac{1}{\Gamma(\nu)} \int_a^{\sigma(t-\nu h)} (t - \sigma_h(s))_h^{(\nu-1)} u(s) \Delta_h s \\ &= \frac{1}{\Gamma(\nu)} \sum_{k=a/h}^{t/h-\nu} (t - \sigma_h(kh))_h^{(\nu-1)} u(kh)h. \end{aligned} \tag{19} $$

the delta right $h-$fractional sum of order $\nu > 0$ for $u :_{b,h} \mathbb{N} \to R$ is given by

$$ \begin{aligned} ({}_h\Delta_b^{-\nu} u)(t) &= \frac{1}{\Gamma(\nu)} \int_{\rho(t+\nu h)}^{b} (\rho_h(s) - t)_h^{(\nu-1)} u(s) \nabla_h s \\ &= \frac{1}{\Gamma(\nu)} \sum_{k=t/h+\nu}^{b/h} (kh - \sigma_h(t))_h^{(\nu-1)} u(kh)h. \end{aligned} \tag{20} $$

Definition 14. (Delta $h-RL$ fractional differences) The delta left $h-$fractional difference operator of order $\nu > 0$ has the form

$$ \begin{aligned} ({}_a\Delta_h^{\nu} u)(t) &= (\Delta_h^n {}_a\Delta_h^{-(n-\nu)} u)(t) \\ &= \frac{\Delta_h^n}{\Gamma(n-\nu)} \sum_{k=a/h}^{t/h-(n-\nu)} (t - \sigma_h(kh))_h^{(n-\nu-1)} u(kh)h, \quad t \in \mathbb{N}_{a+(n-\nu)h,h} \end{aligned} \tag{21} $$

and the delta right $h-$fractional differences of order $\nu > 0$ is defined as

$$ \begin{aligned} ({}_h\Delta_b^{\nu} u)(t) &= (-1)^n (\nabla_h^n {}_h\Delta_b^{-(n-\nu)} u)(t) \\ &= \frac{(-1)^n \nabla_h^n}{\Gamma(n-\nu)} \sum_{k=t/h+n-\nu}^{b/h} (kh - \sigma_h(t))_h^{(n-\nu-1)} u(kh)h \quad t \in_{b-(n-\nu)h,h} \mathbb{N} \end{aligned} \tag{22} $$

Definition 15. (Caputo $h-$fractional differences) Let u be defined on $\mathbb{N}_{a,h}$ and ${}_{b,h}\mathbb{N}$ respectively. Then the left and right delta Caputo $h-$fractional differences of order $\nu > 0$ are defined by

$$ ({}_a^C\Delta_h^{\nu} u)(t) = (\Delta_h^n {}_a\Delta_h^{-(n-\nu)} u)(t), \quad t \in \mathbb{N}_{a+(n-\nu)h,h}, \tag{23} $$

$$ ({}_h^C\Delta_b^{\nu} u)(t) = (-1)^n (\nabla_h^n {}_h\Delta_b^{-(n-\nu)} u)(t), \quad t \in {}_{b-(n-\nu)h,h}\mathbb{N}, \tag{24} $$

where $n = [\nu] + 1$.

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Lemma 16. [1] *Let* $\nu, \mu, h > 0$. *Then*

$$ {}_{a+\mu h}\Delta_h^{-\nu}(t-a)_h^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu+\nu)}(t-a)_h^{(\nu+\mu-1)}, \tag{25} $$

$$ {}_h\Delta_{b-\mu h}^{-\nu}(b-t)_h^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu+\nu)}(b-t)_h^{(\nu+\mu-1)}. \tag{26} $$

3. Generalized delta discrete $h-$ Laplace transform and its convolution

Following the time scale calculus, we have the following definition for the delta discrete Laplace transform on $\mathbb{N}_{a,h}$.

Definition 17. Assume that $u(t)$ is defined on $\mathbb{N}_{a,h}$. Then, the generalized delta discrete Laplace transform of u is defined by

$$ \mathcal{L}_{a,h}\{u(t)\}(s) = \int_a^\infty {}_h\tilde{e}^{\sigma}_{\ominus s}(t,a)u(t)\Delta_h t $$

$$ = \int_a^\infty \frac{{}_h\tilde{e}_{\ominus s}(t,a)}{1+hs}u(t)\Delta_h t = \int_a^\infty \left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}+1} u(t)\Delta_h t. \tag{27} $$

Using the closed and summation form solution, the above equation can be written as

$$ \mathcal{L}_{a,h}\{u(t)\}(s) = {}_a\Delta_h^{-1}u(t)\left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}+1}\Big|_a^\infty = h\sum_{i=a/h}^{\infty} u(ih)\left(\frac{1}{1+hs}\right)^{i-\frac{a}{h}+1}. \tag{28} $$

Remark 18. (i) In case $a=0$, we get

$$ \mathcal{L}_h\{u(t)\}(s) = \Delta_h^{-1}u(t)\left(\frac{1}{1+hs}\right)^{\frac{t}{h}+1}\Big|_0^\infty = h\sum_{i=0}^{\infty} u(ih)\left(\frac{1}{1+hs}\right)^{i+1}. \tag{29} $$

(ii) In a special case $h=1$, we have

$$ \mathcal{L}_a\{u(t)\}(s) = {}_a\Delta^{-1}u(t)\left(\frac{1}{1+s}\right)^{t-a+1}\Big|_a^\infty = \sum_{i=a}^{\infty} u(i)\left(\frac{1}{1+s}\right)^{i-a+1}, \tag{30} $$

which is existing in[12].

Theorem 19. *Let* $t \in \mathbb{N}_{a,h}$, $h, \nu > 0$ *and* $hs \neq -1$ *then we have*

$$ {}_a\Delta_h^{-1}\left[t_h^{(\mu)}\left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}+1}\right] = \sum_{i=1}^{\mu+1} \frac{(-1)\mu^{(i-1)}t_h^{(\mu+1-i)}\left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}}}{s^i}. \tag{31} $$

Proof. Taking $u(t) = t_h^{(1)}$ and $v(t) = \left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}+1}$ in (7), using (6) and (11), we get

$$ {}_a\Delta_h^{-1}\left[t_h^{(1)}\left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}+1}\right] = -\frac{t_h^{(1)}\left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}}}{s} - \frac{\left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}}}{s^2}. \tag{32} $$

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Again taking $u(t) = t_h^{(2)}$ and $v(t) = \left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}+1}$ in (7), gives

$$
{}_a\Delta_h^{-1}\left[t_h^{(2)}\left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}+1}\right] = -\frac{t_h^{(2)}\left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}}}{s} + \frac{2}{s}\,{}_a\Delta_h^{-1}\left[t_h^{(1)}\left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}+1}\right]. \tag{33}
$$

Now applying (32) and simplifying we get

$$
{}_a\Delta_h^{-1}\left[t_h^{(2)}\left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}+1}\right] = -\frac{t_h^{(2)}\left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}}}{s} - \frac{2t_h^{(1)}\left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}}}{s^2} - \frac{2t_h^{(1)}\left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}}}{s^3}, \tag{34}
$$

which can be rewritten as follows,

$$
{}_a\Delta_h^{-1}\left[t_h^{(2)}\left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}+1}\right] = \sum_{i=1}^{3}\frac{(-1)2^{(i-1)}t_h^{(3-i)}\left(\frac{1}{1+hs}\right)^{\frac{t-a}{h}}}{s^i}. \tag{35}
$$

By continuing the above process upto μ times, we get (31). $\square$

Lemma 20. *Let $\mu, h > 0$ and $s \neq 0$, then*

$$
\mathcal{L}_h\{t_h^{(\mu)}\}(s) = \frac{\mu!}{s^{\mu+1}}. \tag{36}
$$

Proof. The proof follows by using (29) by applying the limits 0 to ∞ in (31). $\square$

Example 21. By equating (29) and (36), we get

$$
\mathcal{L}_h\{t_h^{(\mu)}\}(s) = h\sum_{i=0}^{\infty}(ih)_h^{(\mu)}\left(\frac{1}{1+hs}\right)^{i+1} = \frac{\mu!}{s^{\mu+1}}. \tag{37}
$$

Which is verified by MATLAB for the particular values $h = 3, \mu = 2$ and $s = 10$ has equal in both closed and summation form solution numerically 0.0020.

The solution of input function(signal) is analyzed graphically as follows:

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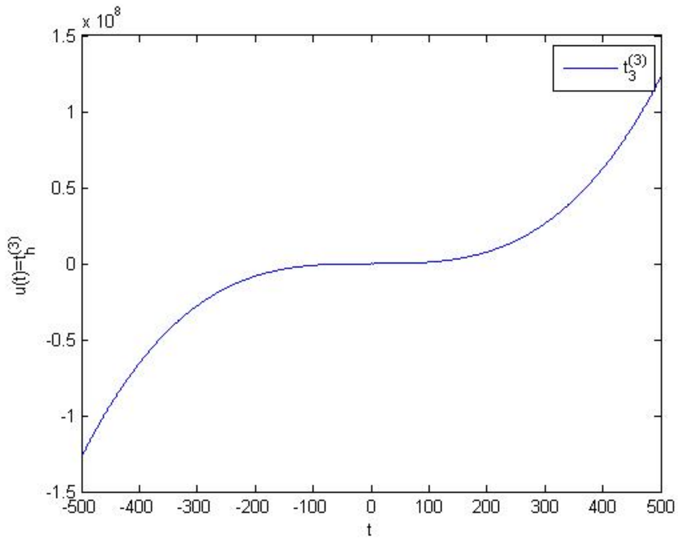

Figure 1. Time Domain(t)

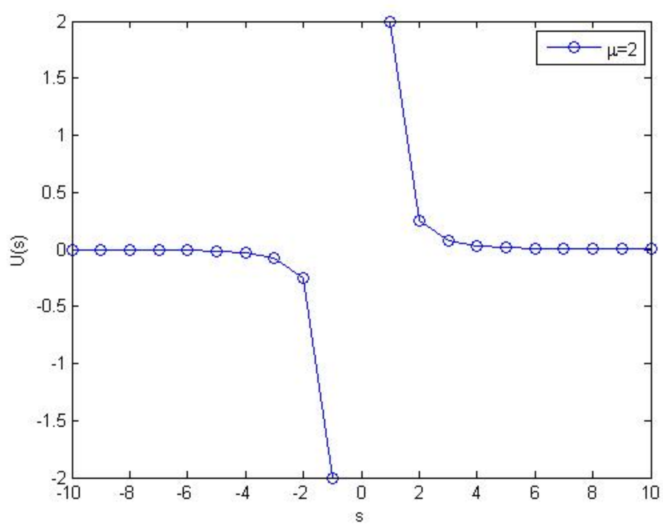

Figure 2. Frequency(s)

Figure 1. is a plot of the input function(signal) $t_3^{(3)}$ and Figure 2. is a plot of output function in the frequency domain for $\mu = 2$ it tells that for the converges the condition is $s \neq 0$.

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Remark 22. By the virtue of (36), we can write the Laplace transform of $t_h^{(\nu-1)}$ for fraction $\nu > 0$ as follows

$$\mathcal{L}_h\{t_h^{(\nu-1)}\}(s) = \frac{\Gamma(\nu)}{s^\nu}. \tag{38}$$

Definition 23. Let $s \in \mathbb{R}, 0 < \nu < 1$ and $u, v : \mathbb{N}_{a,h} \to \mathbb{R}$ be the functions. Then the delta $h-$discrete convolution of u and v is defined by

$$(u * v)(t) = \int_a^t v(t - \sigma_h(s) + a)u(s)\Delta_h s = h \sum_{i=a/h}^{t/h-1} u(ih)v(t - \sigma_h(ih) + a),\ t \in \mathbb{N}_{a,h} \tag{39}$$

Theorem 24. *(The $h-$convolution theorem) For $s \in \mathbb{R}$ and u, v defined on $\mathbb{N}_{a,h}$, we have*

$$\mathcal{L}_{a,h}\{(u * v)(t)\}(s) = \mathcal{L}_{a,h}\{u(t)\}(s)\mathcal{L}_{a,h}\{v(t)\}(s). \tag{40}$$

Proof. By the definition, we have

$$\begin{aligned}
\mathcal{L}_{a,h}\{(u * v)(t)\}(s) &= h \sum_{i=a/h}^{\infty} \left(\frac{1}{1+hs}\right)^{i-\frac{a}{h}+1} h \sum_{k=a/h}^{i-1} u(kh)v(ih - \sigma_h(kh) + a) \\
&= h^2 \sum_{k=a/h}^{\infty} \sum_{i=k+1}^{\infty} \frac{u(kh)v((i-k-1+a/h)h)}{(1+hs)^{i-a/h+1}} \\
&= h^2 \sum_{k=a/h}^{\infty} \sum_{j=a/h}^{\infty} \frac{u(kh)v(jh)}{(1+hs)^{j+k+1-a/h-a/h+1}} \\
&= h \sum_{k=a/h}^{\infty} \frac{u(kh)}{(1+hs)^{k-a/h+1}} h \sum_{j=a/h}^{\infty} \frac{v(jh)}{(1+hs)^{j-a/h+1}} \\
&= \mathcal{L}_{a,h}\{u(t)\}(s)\mathcal{L}_{a,h}\{v(t)\}(s),
\end{aligned}$$

where the change of variable $j = i - k - 1 + a/h$ is used. □

The following lemma is a generalization of the result in [12] to $h\mathbb{Z}$.

Lemma 25. *Let u be defined on $\mathbb{N}_{a,h}$. Then,*

$$\mathcal{L}_{a,h}\{\Delta_h u(t)\}(s) = s\mathcal{F}_a(s) - u(a), \tag{41}$$

where $\mathcal{F}_a(s) = \mathcal{L}_{a,h}\{u(t)\}(s)$.

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Proof. By Definition 17 and using (10), we see that

$$\begin{aligned}
\mathcal{L}_{a,h}\{\Delta_h u(t)\}(s) &= \int_a^\infty {}_h\tilde{e}^\sigma_{\ominus s}(t,a)\Delta_h u(t)\Delta_h t \\
&= {}_h\tilde{e}^\sigma_{\ominus s}(t,a)u(t)\Big|_a^\infty - \int_a^\infty \ominus s_h \tilde{e}_{\ominus s}(t,a)u(t)\Delta_h t \\
&= -u(a) + s\int_a^\infty {}_h\tilde{e}^\sigma_{\ominus s}(t,a)u(t)\Delta_h t \\
&= s\mathcal{F}_a(s) - u(a).
\end{aligned}$$

□

4. Delta fractional proportional differences on $h\mathbb{Z}$

From the time scale calculus in view of [10] and [2], the delta exponential function $\tilde{e}_p(t,a) = (1+ph)^{\frac{t-a}{h}}$, where $p(t) = \dfrac{\rho-1}{\rho}$ on $h\mathbb{Z}$ satisfies the difference equation

$$\Delta_h u(t) = \frac{\rho-1}{\rho}u(t),\ u(a) = 1,$$

where $\rho \neq \dfrac{h}{h+1}$. Notice that $h=1$ implies that the proportional factor $\rho \neq \dfrac{1}{2}$.

The delta $h-$discrete proportional differences of order $0<\rho\leq 1$ for function u on $\mathbb{N}_{a,h}$ is given by

$$(\Delta_h^\rho u)(t) = (1-\rho)u(t) + \rho\Delta_h u(t),\ t\in\mathbb{N}_{a+h,h}$$

where the regressivity condition insists that $\rho \neq \dfrac{h}{h+1}$.

Remark 26. [10] Since, the first order proportional type difference equation

$$(\Delta_h^\rho v)(t) = (1-\rho)v(t) + \rho\Delta_h v(t) = u(t),\ v(a) = v_a$$

has the solution $v(t) = v_a\tilde{e}_p(t,a) + \dfrac{1}{\rho}\int_a^t \tilde{e}_p(t-s-h,0)u(s)\Delta_h s$, where $p = \dfrac{\rho-1}{\rho}$ and $\int_a^t u(s)\Delta_h s = h\sum_{i=a/h}^{t/h-1} u(ih)$.

Definition 27. For $0<\rho\leq 1$, the first order proportional sum with Δ^ρ is defined by

$$({}_a\Delta_h^{-1,\rho}u)(t) = \frac{1}{\rho}\int_a^t \tilde{e}_p(t-s-h,0)u(s)\Delta_h s,\ p = \frac{\rho-1}{\rho}. \tag{42}$$

Remark 28. By using the definition of convolution the above relation can be written as $({}_a\Delta_h^{-1,\rho}u)(t) = \dfrac{1}{\rho}(u*\tilde{e}_p(t-a),0)$.

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Property 29. Let u is defined on $\mathbb{N}_{a,h}$, we have

$$({}_a\Delta_h^{-1,\rho}\Delta_h^{\rho}u)(t) = u(t) - \tilde{e}_p(t-a,0)u(a) = u(t) - \left(\frac{\rho+(\rho-1)h}{\rho}\right)^{\frac{t-a}{h}} u(a). \tag{43}$$

Proof. Applying $h-$Laplace transform on ${}_a\Delta_h^{-1,\rho}\Delta_h^{\rho}u(t)$, we see that

$$\begin{aligned}
\mathcal{L}_{a,h}\{{}_a\Delta_h^{-1,\rho}\Delta_h^{\rho}u(t)\}(s) &= \mathcal{L}_{a,h}\{\frac{1}{\rho}\Delta_h^{\rho}u(t) * \tilde{e}_p(t-a,0)\}(s) \\
&= \frac{1}{\rho}\mathcal{L}_{a,h}\{(1-\rho)u(t)+\rho\Delta_h u(t)\}(s)\mathcal{L}_{a,h}\{\tilde{e}_p(t-a,0)\}(s) \\
&= \left\{\frac{1-\rho}{\rho}\mathcal{F}_a(s) + s\mathcal{F}_a(s) - u(a)\right\}\left(\frac{\rho}{s\rho-\rho+1}\right) \\
&= \mathcal{F}_a(s) - u(a)\frac{\rho}{s\rho-\rho+1}.
\end{aligned}$$

Applying inverse Laplace transform $\mathcal{L}_{a,h}^{-1}$, we obtain (30). ☐

To generalize the first order proportional difference to fractional sum, we see that

$$\begin{aligned}
({}_a\Delta_h^{-n,\rho}u)(t) = \frac{1}{\rho}\int_a^t &\tilde{e}_p(t-\psi_1-h,0)\Delta_h\psi_1 \\
\frac{1}{\rho}\int_a^{\psi_1} &\tilde{e}_p(\psi_1-\psi_2-h,0)\Delta_h\psi_2 \\
\dots\frac{1}{\rho}\int_a^{\psi_{n-1}} &\tilde{e}_p(\psi_{n-1}-\psi_n-h,0)u(s)\Delta_h\psi_n
\end{aligned}$$

$$({}_a\Delta_h^{-n,\rho}u)(t) = \frac{1}{\rho^n\Gamma(n)}\int_a^t \tilde{e}_p(t-\psi-nh,0)(t-\sigma_h(\psi))_h^{(n-1)}u(\psi)\Delta_h\psi,$$

where the Lemma 9 is used. By the virtue of the above result, we can present the following generalized type fractional proportional sum.

Definition 30. For $0<\rho\le 1$ and $\nu>0$, we define the left proportional fractional sum of u by

$$\begin{aligned}
({}_a\Delta_h^{-\nu,\rho}u)(t) &= \frac{1}{\rho^\nu\Gamma(\nu)}\int_a^t \tilde{e}_p(t-\psi-\nu h,0)(t-\sigma_h(\psi))_h^{(\nu-1)}u(\psi)\Delta_h\psi, \\
&= \frac{h}{\rho^\nu\Gamma(\nu)}\sum_{i=a/h}^{t/h-\nu}\tilde{e}_p(t-(i+\nu)h,0)(t-\sigma_h(ih))_h^{(\nu-1)}u(ih).
\end{aligned} \tag{44}$$

and the right proportional fractional sum as

$$({}_h\Delta_b^{-\nu,\rho}u)(t) = \frac{1}{\rho^\nu\Gamma(\nu)}\int_t^b \tilde{e}_p(\psi-\nu h-t,0)(\rho_h(\psi)-t)_h^{(\nu-1)}u(\psi)\Delta_h\psi,$$

$$= \frac{h}{\rho^{\nu}\Gamma(\nu)} \sum_{i=t/h+\nu}^{b/h} \tilde{e}_p((i-\nu)h - t, 0)(\rho_h(ih) - t)_h^{(\nu-1)} u(ih). \tag{45}$$

Remark 31. Using convolution, we can express the left proportional fractional sum as

$$({}_a\Delta_h^{-\nu,\rho} u)(t) = \frac{1}{\rho^{\nu}\Gamma(\nu)} \left(\frac{\rho + (\rho-1)h}{\rho}\right)^{(1-\nu)} \tilde{e}_p(t,0) t_h^{(\nu-1)} * u(t).$$

In particular case $h = 1$,

$$({}_a\Delta^{-\nu,\rho} u)(t) = \frac{1}{\rho^{\nu}\Gamma(\nu)} \left(\frac{2\rho - 1}{\rho}\right)^{(1-\nu)} \tilde{e}_p(t,0) t^{(\nu-1)} * u(t).$$

Definition 32. (RL-fractional proportional) For $0 < \rho \leq 1$ and $\nu > 0$, we define the left proportional fractional sum of u by
$({}_a\Delta_h^{\nu,\rho} u)(t) = (\Delta_h^{n,\rho}\, {}_a\Delta_h^{-(n-\nu),\rho} u)(t)$

$$= \frac{\Delta_h^{n,\rho}}{\rho^{n-\nu}\Gamma(n-\nu)} \int_a^t \tilde{e}_p(t - \psi - (n-\nu)h, 0)(t - \sigma_h(\psi))_h^{(n-\nu-1)} u(\psi)\Delta_h\psi, \tag{46}$$

and the right proportional fractional sum as
$({}_h\Delta_b^{\nu,\rho} u)(t) = (-1)^n (\nabla_h^{n,\rho}\, {}_h\Delta_b^{-(n-\nu),\rho} u)(t)$

$$= \frac{(-1)^n \nabla_h^{n,\rho}}{\rho^{n-\nu}\Gamma(n-\nu)} \int_t^b \tilde{e}_p(\psi - (n-\nu)h - t, 0)(\rho_h(\psi) - t)_h^{(n-\nu-1)} u(\psi)\Delta_h\psi. \tag{47}$$

Remark 33. (i) $\lim_{\nu\to 0}({}_a\Delta_h^{\nu,\rho} u)(t) = u(t)$ and (ii) $\lim_{\nu\to 1}({}_a\Delta_h^{\nu,\rho} u)(t) = ({}_a\Delta_h^{\rho} u)(t)$.

Theorem 34. *(semi group property for delta fractional proportional) Assume that $u(t)$ is defined on $\mathbb{N}_{a,h}$. Then*

$${}_a\Delta_h^{-\mu,\rho}({}_a\Delta_h^{-\nu,\rho} u)(t) = {}_a\Delta_h^{-\nu,\rho}({}_a\Delta_h^{-\mu,\rho} u)(t) = ({}_a\Delta_h^{-(\mu+\nu),\rho} u)(t). \tag{48}$$

Proof. By definition 30 and using the lemma 16, we obtain

$$\begin{aligned}
{}_a\Delta_h^{-\mu,\rho}({}_a\Delta_h^{-\nu,\rho} u)(t) &= \frac{1}{\rho^{\mu+\nu}\Gamma(\mu)\Gamma(\nu)} \int_a^t \int_a^w \tilde{e}_p(t - w - \mu h, 0)\tilde{e}_p(w - \psi - \nu h, 0) \\
&\quad (t - \sigma_h(w))_h^{(\mu-1)} (t - \sigma_h(\psi))_h^{(\nu-1)} u(\psi)\Delta_h\psi\Delta_h w \\
&= \frac{1}{\rho^{\mu+\nu}\Gamma(\nu)} \int_a^t \tilde{e}_p(t - \psi - (\mu+\nu)h, 0) u(\psi) \\
&\quad \frac{1}{\Gamma(\mu)} \int_\psi^t (t-w)_h^{(\mu-1)} (w-\psi)_h^{(\nu-1)} \Delta_h w \Delta_h\psi
\end{aligned}$$

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$$= \frac{\rho^{-(\mu+\nu)}}{\Gamma(\nu)} \int_a^t u(\psi)\tilde{e}_p(t-\psi-(\mu+\nu)h,0)_\psi\Delta_h^{-\mu}(t-\psi)_h^{(\nu-1)}\Delta_h\psi$$
$$= \frac{\rho^{-(\mu+\nu)}}{\Gamma(\mu+\nu)} \int_a^t u(\psi)\tilde{e}_p(t-\psi-(\mu+\nu)h,0)(t-\psi)_h^{(\mu+\nu-1)}\Delta_h\psi$$
$$= ({}_a\Delta_h^{-(\mu+\nu),\rho}u)(t).$$

□

Corollary 2. *Let u is defined on $\mathbb{N}_{a,h}$. Then*

$$ {}_a\Delta_h^{\mu,\rho}({}_a\Delta_h^{-\nu,\rho}u)(t) = ({}_a\Delta_h^{-(\nu-\mu),\rho}u)(t) \tag{49}$$

Proof. The proof follows by taking negative index of μ in theorem 34. □

The following theorems is the Laplace transform of fractional proportional type difference equation and fractional proportional sum.

Theorem 35. *Assume that u is defined on $\mathbb{N}_{a,h}$. Then*

$$\mathcal{L}_{a,h}\{{}_a\Delta_h^{-n,\rho}u(t)\}(s) = (\rho s+1-\rho)^n\mathcal{L}_{a,h}\{u(t)\}(s) - \rho\sum_{r=0}^{n-1}(\rho s+1-\rho)^{n-1-r}(\Delta_h^{r,\rho}u)(a) \tag{50}$$

Proof. By applying $h-$Laplace transform on remark 26 and using lemma 25, we arrive

$$\begin{aligned}\mathcal{L}_{a,h}\{{}_a\Delta_h^{1,\rho}u(t)\}(s) &= \mathcal{L}_{a,h}\{(1-\rho)u(t)+\rho\Delta_h u(t)\}(s)\\ &= (1-\rho)\mathcal{L}_{a,h}\{u(t)\}(s) + \rho\mathcal{L}_{a,h}\{{}_a\Delta_h u(t)\}(s)\\ &= (1-\rho)\mathcal{L}_{a,h}\{u(t)\}(s) + \rho\{s\mathcal{L}_{a,h}\{u(t)\}(s) - u(a)\}\\ &= (1-\rho+\rho s)\mathcal{L}_{a,h}\{u(t)\}(s) - \rho u(a)\end{aligned}$$

By applying induction on n, we get (50). □

Theorem 36. *For $\nu \in R^+$, $u(t)$ be the function, then we have*

$$\mathcal{L}_{a,h}\{{}_a\Delta_h^{-\nu,\rho}u(t)\}(s) = \left(\frac{\rho+(\rho-1)h}{\rho}\right)^{(1-\nu)}\frac{\mathcal{L}_{a,h}\{u(t)\}(s)}{(\rho s-\rho+1)^\nu} \tag{51}$$

Proof. By the definition of Laplace transform and convolution, we get

$$\begin{aligned}\mathcal{L}_{a,h}\{{}_a\Delta_h^{-\nu,\rho}u(t)\}(s) &= \frac{1}{\rho^\nu\Gamma(\nu)}\left(\frac{\rho+(\rho-1)h}{\rho}\right)^{(1-\nu)}\mathcal{L}_{a,h}\{\tilde{e}_p(t,0)t_h^{(\nu-1)} * u(t)\}(s)\\ &= \frac{1}{\rho^\nu\Gamma(\nu)}\left(\frac{\rho+(\rho-1)h}{\rho}\right)^{(1-\nu)}\mathcal{L}_{a,h}\{\tilde{e}_p(t,0)t_h^{(\nu-1)}\}(s)\mathcal{L}_{a,h}\{u(t)\}(s)\\ &= \frac{1}{\rho^\nu\Gamma(\nu)}\left(\frac{\rho+(\rho-1)h}{\rho}\right)^{(1-\nu)}\frac{\Gamma(\nu)}{\left(s-\frac{\rho-1}{\rho}\right)^\nu}\mathcal{L}_{a,h}\{u(t)\}(s),\end{aligned}$$

which completes the proof. □

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5. Conclusion

This present work focuses on delta type fractional sums and differences, a new type of generalized discrete $h-$Laplace transform and its convolution are defined. Several properties and results of $h-$discrete Laplace transform have been discussed. Also, the researchers investigates the newly proposed delta type fractional proportional sums and differences on discrete RL and Mittag-Leffler functions. The findings are verified and analysed by MATLAB. Through this paper, when $h = 1$ and $\rho = 1$, we get the existing results in the literature.

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