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HOMOMORPHISMS AND DERIVATIONS OF A GENERALIZED ADDITIVE FUNCTIONAL EQUATION

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Abstract. In this paper, we introduce and investigate the generalized Ulam - Hyers stability of homomorphisms and derivations of a generalized additive functional equation in Banach, Quasi - Banach, C^*, Lie C^*, Jordan C^* Algebras.

1. Introduction

In Ulam [26] proposed the general Ulam stability problem: When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true? In Hyers [7] gave the first affirmative answer to the question of Ulam for additive functional equations on Banach spaces. Hyers result has since then seen many significant generalizations, both in terms of the control condition used to define the concept of approximate solution one can see [2, 5, 17, 21, 23].

One of the most famous functional equation is the **additive functional equation**

$$\lambda(u+v) = \lambda(u) + \lambda(v) \tag{1}$$

having solution $\lambda(u) = cu$. This functional equation was first treated by A.M. Legendre (1791) and C.F. Gauss (1809). In 1821, it was first solved by A.L. Cauchy in the class of continuous real-valued functions. It is often called an Cauchy additive functional equation in honor of A.L. Cauchy [1, 8].

In this paper, we introduce and investigate the generalized Ulam - Hyers stability of homomorphisms and derivations of a generalized additive functional equation

$$\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) = (\alpha + \beta)(\lambda(u) + \lambda(v)) \tag{2}$$

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where $\alpha, \beta \neq 0$ in Banach, Quasi - Banach, C^*, Lie C^*, Jordan C^*, Algebras.

Now, we provide the general solution of the functional equation (2).

Theorem 1. *Assume V_1 and V_2 are real vector spaces. Suppose that $\lambda : V_1 \to V_2$ satisfies the functional equation (1) then $\lambda : V_1 \to V_2$ satisfies the functional equation (2).*

During the last seven decades the stability problems of various functional equations in several algebras have been broadly investigated by number of mathematicians and more detail's about the definitions on all the algebras see [3, 4, 6, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20, 22, 24, 25]. In each sections, we give basic definitions about algebras and prove the generalized Ulam - Hyers stability of homomorphisms and derivations with respect to that algebras.

2. Stability Results in Banach Algebras

2.1. Banach Algebra Definitions.

Definition 2. A complex Banach space A is said to be a **Banach algebra** if it satisfies the condition

$$||xy|| \leq C||x||||y||$$

for all $x, y \in A$.

Definition 3. Let A and B be real Banach algebras. A mapping $H : A \to B$ is called a **algebra homomorphism** if

$$H(xy) = H(x)H(y)$$

for all $x, y \in A$.

Definition 4. Let A and B be real Banach algebras. A $D : A \to A$ is called a **algebra derivation** if

$$D(xy) = D(x)y + xD(y)$$

for all $x, y \in A$.

In order to establish the stability results, throughout this section let us assume $\mathcal{A}$ is a Banach algebra with norm $|| \cdot ||_A$ and $\mathcal{B}$ is a Banach algebra with norm $|| \cdot ||_B$.

2.2. Homomorphism Stability Result.

Theorem 5. *If $\lambda : \mathcal{A} \to \mathcal{B}$ and $\eta : \mathcal{A}^2 \to [0, \infty)$ are functions satisfying the double inequalities*

$$\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \leq \eta(u, v) \tag{1}$$

$$\|\lambda(uv) - \lambda(u) - \lambda(v)\|_{\mathcal{B}} \leq \eta(u, v) \tag{2}$$

and

$$\lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) = 0 = \lim_{\delta\to\infty} \frac{1}{\gamma^{2\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) \tag{3}$$

for all $u, v \in \mathcal{A}$ where

$$\nu = \pm 1 \qquad \text{and} \qquad \gamma = \alpha + \beta. \tag{4}$$

Then there exists a unique homomorphism function $\mathcal{H}:\mathcal{A}\to\mathcal{B}$ satisfying the functional equation (2) and

$$\|\lambda(u)-\mathcal{H}(u)\|_{\mathcal{B}} \le \frac{1}{2\gamma}\sum_{\chi=\frac{1-\nu}{2}}^{\infty}\frac{\eta(\gamma^{\chi\nu}u,\gamma^{\chi\nu}u)}{\gamma^{\chi\nu}} \tag{5}$$

and the mapping $\mathcal{H}(u)$ is defined by

$$\mathcal{H}(u)=\lim_{\delta\to\infty}\frac{1}{\gamma^{\delta\nu}}\lambda(\gamma^{\delta\nu}u) \tag{6}$$

for all $u\in\mathcal{A}$.

Proof. Assume $\nu=1$. Letting (u,v) by (u,u) in (1), we arrive

$$\|2\lambda((\alpha+\beta)u)-2(\alpha+\beta)\lambda(u)\|_{\mathcal{B}}\le\eta(u,u)\Longrightarrow\|\lambda(\gamma u)-\gamma\lambda(u)\|_{\mathcal{B}}\le\frac{1}{2}\eta(u,u) \tag{7}$$

for all $u\in\mathcal{A}$. It follows from above inequality that

$$\left\|\frac{\lambda(\gamma u)}{\gamma}-\lambda(u)\right\|_{\mathcal{B}}\le\frac{1}{2\gamma}\eta(u,u) \tag{8}$$

for all $u\in\mathcal{A}$. Now replacing u by γu and dividing by γ in (8), we obtain

$$\left\|\frac{\lambda(\gamma^2u)}{\gamma^2}-\frac{\lambda(\gamma u)}{\gamma}\right\|_{\mathcal{B}}\le\frac{1}{2\gamma^2}\eta(\gamma u,\gamma u) \tag{9}$$

for all $u\in\mathcal{A}$. From (8) and (9), we get

$$\left\|\frac{\lambda(\gamma^2u)}{\gamma^2}-\lambda(u)\right\|_{\mathcal{B}}\le\frac{1}{2\gamma}\left[\eta(u,u)+\frac{\eta(\gamma u,\gamma u)}{\gamma}\right] \tag{10}$$

for all $u\in\mathcal{A}$. Proceeding further and using induction on a positive integer δ, we have

$$\left\|\frac{\lambda(\gamma^\delta u)}{\gamma^\delta}-\lambda(u)\right\|_{\mathcal{B}}\le\frac{1}{2\gamma}\sum_{\chi=0}^{\delta-1}\frac{1}{\gamma^\chi}\eta(\gamma^\chi u,\gamma^\chi u) \tag{11}$$

for all $u\in\mathcal{A}$. It is easy to verify that the sequence

$$\left\{\frac{\lambda(\gamma^\delta u)}{\gamma^\delta}\right\},$$

is a Cauchy sequence by replacing u by $\gamma^\epsilon u$ and dividing by γ^ϵ in (11), for any $\epsilon,\delta>0$. Since $\mathcal{B}$ is complete, there exists a mapping $\mathcal{H}(u):\mathcal{A}\to\mathcal{B}$ such that

$$\mathcal{H}(u)=\lim_{\delta\to\infty}\frac{1}{\gamma^\delta}\lambda(\gamma^\delta u),\qquad for\quad all\qquad u\in\mathcal{A}.$$

Letting $\delta\to\infty$ in (11), we see that (5) holds for all $u\in\mathcal{A}$. To show that $\mathcal{H}$ satisfies (2), replacing (u,v) by $(\gamma^\delta u,\gamma^\delta v)$ and dividing by γ^δ in (1), we obtain

$$\frac{1}{\gamma^\delta}\left\|\lambda(\gamma^\delta(\alpha u+\beta v))+\lambda(\gamma^\delta(\beta u+\alpha v))-(\alpha+\beta)(\lambda(\gamma^\delta u)+\lambda(\gamma^\delta v))\right\|_{\mathcal{B}}\frac{1}{\gamma^n}\eta(\gamma^\delta u,\gamma^\delta v)$$

for all $u,v\in\mathcal{A}$. Letting $\delta\to\infty$ in the above inequality and using the definition of $\mathcal{H}(u)$, we see that

$$\mathcal{H}(\alpha u+\beta v)+\mathcal{H}(\beta u+\alpha v)=(\alpha+\beta)(\mathcal{H}(u)+\mathcal{H}(v)).$$

Thus the existence of $\mathcal{H}$ satisfies the additive functional equation (2) for all $u, v \in \mathcal{A}$.

From (2) and definition of $\mathcal{H}$, we achieve

$$\begin{aligned}\|\mathcal{H}(uv) - \mathcal{H}(u)\mathcal{H}(v)\|_{\mathcal{B}} &= \frac{1}{\gamma^{2\delta}} \left\|\lambda(\gamma^{\delta}u\ \gamma^{\delta}v) - \lambda(\gamma^{\delta}u)\lambda(\gamma^{\delta}v)\right\|_{\mathcal{B}} \\ &\leq \frac{1}{\gamma^{2\delta}}\eta(\gamma^{\delta}u, \gamma^{\delta}v) \to 0 \quad as \ \ \delta \to \infty.\end{aligned}$$

Therefore

$$\mathcal{H}(uv) = \mathcal{H}(u)\mathcal{H}(v)$$

for all $u, v \in \mathcal{A}$. Thus, $\mathcal{H}$ is a algebra homomorphism. To prove existence of $\mathcal{H}$ is unique, we assume $\mathcal{H}'(u)$ be another homomorphism mapping satisfying (2) and (4), then

$$\begin{aligned}\|\mathcal{H}(u) - \mathcal{H}'(u)\|_{\mathcal{B}} &= \frac{1}{\gamma^{\epsilon}} \|\mathcal{H}(\gamma^{\epsilon}u) - \mathcal{H}'(\gamma^{\epsilon}u)\|_{\mathcal{B}} \\ &\leq \frac{1}{\gamma^{\epsilon}} \left\{\|\mathcal{H}(\gamma^{\epsilon}u) - \lambda(\gamma^{\epsilon}u)\|_{\mathcal{B}} + \|\lambda(\gamma^{n}u) - \mathcal{H}'(\gamma^{n}u)\|_{\mathcal{B}}\right\} \\ &\leq \frac{2}{2\gamma} \sum_{\chi=0}^{\infty} \frac{1}{\gamma^{(\delta+\epsilon)}} \eta(\gamma^{\delta+\epsilon}u, \gamma^{\delta+\epsilon}u) \\ &\to 0 \quad as \ \ \delta \to \infty\end{aligned}$$

for all $u \in \mathcal{A}$. Hence $\mathcal{H}$ is unique. Thus the theorem holds for $\nu = 1$.

Letting u by $\frac{u}{\gamma}$ in (7), we get

$$\left\|\lambda(u) - \gamma\lambda\left(\frac{u}{\gamma}\right)\right\|_{\mathcal{B}} \leq \frac{1}{2}\eta\left(\frac{u}{\gamma}, \frac{u}{\gamma}\right) \tag{12}$$

for all $u \in \mathcal{A}$. Again setting u by $\frac{u}{\gamma}$ and multiply by γ in (12), we obtain

$$\left\|\gamma\lambda\left(\frac{u}{\gamma}\right) - \gamma^2\lambda\left(\frac{u}{\gamma^2}\right)\right\|_{\mathcal{B}} \leq \frac{\gamma}{2}\eta\left(\frac{u}{\gamma^2}, \frac{u}{\gamma^2}\right) \tag{13}$$

for all $u \in \mathcal{A}$. From (12) and (13), we achieve

$$\left\|\lambda(u) - \gamma^2\lambda\left(\frac{u}{\gamma^2}\right)\right\|_{\mathcal{B}} \leq \frac{1}{2}\left[\eta\left(\frac{u}{\gamma}, \frac{u}{\gamma}\right) + \gamma\ \eta\left(\frac{u}{\gamma^2}, \frac{u}{\gamma^2}\right)\right] \tag{14}$$

for all $u \in \mathcal{A}$. Proceeding further and using induction on a positive integer δ, we have

$$\left\|\lambda(u) - \gamma^{\delta}\lambda\left(\frac{u}{\gamma^{\delta}}\right)\right\|_{\mathcal{B}} \leq \frac{1}{2}\sum_{\chi=1}^{\delta} \gamma^{\delta-1}\eta\left(\frac{u}{\gamma^{\delta}}, \frac{u}{\gamma^{\delta}}\right) = \frac{1}{2\gamma}\sum_{\chi=1}^{\delta} \gamma^{\delta}\eta\left(\frac{u}{\gamma^{\delta}}, \frac{u}{\gamma^{\delta}}\right) \tag{15}$$

for all $u \in \mathcal{A}$. The rest of the proof is similar lines to that of case $\nu = 1$. Thus, the theorem holds for $\nu = -1$. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 5 concerning some stabilities of (2).

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Corollary 1. *Suppose $\lambda : \mathcal{A} \to \mathcal{B}$ be a mapping and there exists real numbers π and ϖ such that*

$$\left.\begin{array}{r}\|\lambda(\alpha u+\beta v)+\lambda(\beta u+\alpha v)-(\alpha+\beta)(\lambda(u)+\lambda(v))\|_{\mathcal{B}} \\ \\ \|\lambda(uv)-\lambda(u)-\lambda(v)\|_{\mathcal{B}}\end{array}\right\} \leq \left\{\begin{array}{l}\pi, \\ \pi\left\{||u||_{\mathcal{A}}^{\varpi}+||v||_{\mathcal{A}}^{\varpi}\right\}, \\ \pi\ ||u||_{\mathcal{A}}^{\varpi}||v||_{\mathcal{A}}^{\varpi},\end{array}\right. \tag{16}$$

for all $u, v \in \mathcal{A}$. Then there exists a unique homomorphism function $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ satisfying the functional equation (2) and

$$\|\lambda(u)-\mathcal{H}(u)\|_{\mathcal{B}} \leq \begin{cases} \dfrac{\pi}{2|1-\gamma|}, & \\ \dfrac{\pi||u||_{\mathcal{A}}^{\varpi}}{2|\gamma-\gamma^{\varpi}|}, & \varpi \neq 1 \\ \dfrac{\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma-\gamma^{2\varpi}|}, & 2\varpi \neq 1 \end{cases} \tag{17}$$

for all $u \in \mathcal{A}$.

2.3. **Derivation Stability Result.**

Theorem 6. *If $\lambda : \mathcal{A} \to \mathcal{A}$ and $\eta : \mathcal{A}^2 \to [0, \infty)$ are functions satisfying the double inequalities*

$$\|\lambda(\alpha u+\beta v)+\lambda(\beta u+\alpha v)-(\alpha+\beta)(\lambda(u)+\lambda(v))\|_{\mathcal{B}} \leq \eta(u, v) \tag{18}$$

$$\|\lambda(uv)-u\lambda(v)-\lambda(u)v\|_{\mathcal{B}} \leq \eta(u, v) \tag{19}$$

and

$$\lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \eta(\gamma^{\delta\nu}u, \gamma^{\delta\nu}v) = 0 = \lim_{\delta\to\infty} \frac{1}{\gamma^{2\delta\nu}} \eta(\gamma^{\delta\nu}u, \gamma^{\delta\nu}v) \tag{20}$$

for all $u, v \in \mathcal{A}$ where

$$\nu = \pm 1 \qquad \text{and} \qquad \gamma = \alpha + \beta. \tag{21}$$

Then there exists a unique derivation function $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying the functional equation (2) and

$$\|\lambda(u)-\mathcal{D}(u)\|_{\mathcal{B}} \leq \frac{1}{2\gamma} \sum_{\chi=\frac{1-\nu}{2}}^{\infty} \frac{\eta(\gamma^{\chi\nu}u, \gamma^{\chi\nu}u)}{\gamma^{\chi\nu}} \tag{22}$$

and the mapping $\mathcal{D}(u)$ is defined by

$$\mathcal{D}(u) = \lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \lambda(\gamma^{\delta\nu}u) \tag{23}$$

for all $u \in \mathcal{A}$.

Proof. By the proof of Theorem 5, there exists a unique additive mapping $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying (22). Also, the mapping $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ given by

$$\mathcal{D}(u) = \lim_{\delta\to\infty} \frac{1}{\gamma^{\delta}} \lambda(\gamma^{\delta}u)$$

for all $u \in \mathcal{A}$.

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From (19) and by definition of $\mathcal{D}$, we achieve

$$\begin{aligned}\|\mathcal{D}(uv) - u\mathcal{D}(v) - \mathcal{D}(u)v\|_{\mathcal{B}} &= \frac{1}{\gamma^{2\delta}} \left\|\lambda(\gamma^{\delta}u\ \gamma^{\delta}v) - \gamma^{\delta}u\lambda(\gamma^{\delta}v) - \lambda(\gamma^{\delta}u)\gamma^{\delta}v\right\|_{\mathcal{B}} \\ &\leq \frac{1}{\gamma^{2\delta}}\eta(\gamma^{\delta}u, \gamma^{\delta}v) \to 0 \quad as\ \ \delta \to \infty.\end{aligned}$$

Therefore

$$\mathcal{D}(uv) = u\mathcal{D}(v) + \mathcal{D}(u)v$$

for all $u, v \in \mathcal{A}$. Thus, $\mathcal{D}$ is a algebra derivation. □

The following corollary is an immediate consequence of Theorem 6 concerning some stabilities of (2).

Corollary 2. *Suppose $\lambda : \mathcal{A} \to \mathcal{A}$ be a mapping and there exists real numbers π and ϖ such that*

$$\left.\begin{array}{r}\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \\ \\ \|\lambda(uv) - u\lambda(v) - \lambda(u)v\|_{\mathcal{B}}\end{array}\right\} \leq \left\{\begin{array}{l}\pi, \\ \pi\left\{||u||_{\mathcal{A}}^{\varpi} + ||v||_{\mathcal{A}}^{\varpi}\right\}, \\ \pi\ ||u||_{\mathcal{A}}^{\varpi}||v||_{\mathcal{A}}^{\varpi},\end{array}\right. \tag{24}$$

for all $u, v \in \mathcal{A}$. Then there exists a unique derivation function $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{D}(u)\|_{\mathcal{B}} \leq \left\{\begin{array}{ll}\dfrac{\pi}{2|1-\gamma|}, & \\ \dfrac{\pi||u||_{\mathcal{A}}^{\varpi}}{2|\gamma - \gamma^{\varpi}|}, & \varpi \neq 1 \\ \dfrac{\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1\end{array}\right. \tag{25}$$

for all $u \in \mathcal{A}$.

3. Stability Results in Quasi - Banach Algebras

3.1. Quasi - Banach Algebra Definitions.

Definition 7. Let X be a linear space over $\mathbb{K}$. A quasi norm is a real-valued function on X satisfying the following:

$(QB1)$ $\| x \| \geq 0$ for all $u \in X$ and $\| x \| = 0$ if and only if $u = 0$.
$(QB2)$ $\| \rho x \| = | \rho | . \| x \|$ for all $\rho \in \mathbb{K}$ and all $u \in X$.
$(QB3)$ There is a constant $K \geq 1$ such that $\| x + y \| \leq K (\| x \| + \| y \|)$ for all $u, y \in X$.

The pair $(X, \| \cdot \|)$ is called **quasi normed space** if $\| \cdot \|$ is a quasi norm on X. The smallest possible K is called the modulus of concavity of $\| \cdot \|$.

Definition 8. A **quasi Banach space** is a complete quasi normed space.

Definition 9. A quasi normed space X is called a **quasi normed algebra** if there is a constant C such that

$$||xy|| \leq C||x||||y||$$

for all $u, y \in X$.

Definition 10. Let A and B be quasi normed algebra. A mapping $H : A \to B$ is called a algebra homomorphism if

$$H(xy) = H(x)H(y)$$

for all $u, y \in A$.

Definition 11. Let A and B be quasi normed algebra. A mapping $D : A \to A$ is called a derivation if

$$D(xy) = D(x)y + xD(y)$$

for all $u, y \in A$.

In order to establish the stability results, throughout this section let us assume $\mathcal{A}$ is a quasi norm algebra with norm $|| \cdot ||_A$ and $\mathcal{B}$ is a quasi Banach algebra with norm $|| \cdot ||_B$.

3.2. **Homomorphism Stability Result.**

Theorem 12. *If $\lambda : \mathcal{A} \to \mathcal{B}$ and $\eta : \mathcal{A}^2 \to [0, \infty)$ are functions satisfying the double inequalities*

$$\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \le \eta(u, v) \tag{1}$$

$$\|\lambda(uv) - \lambda(u) - \lambda(v)\|_{\mathcal{B}} \le \eta(u, v) \tag{2}$$

and

$$\lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) = 0 = \lim_{\delta\to\infty} \frac{1}{\gamma^{2\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) \tag{3}$$

for all $u, v \in \mathcal{A}$ where

$$\nu = \pm 1 \qquad and \qquad \gamma = \alpha + \beta. \tag{4}$$

Then there exists a unique homomorphism function $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{H}(u)\|_{\mathcal{B}} \le \frac{K^{\delta-1}}{2\gamma} \sum_{\chi=\frac{1-\nu}{2}}^{\infty} \frac{\eta(\gamma^{\chi\nu} u, \gamma^{\chi\nu} u)}{\gamma^{\chi\nu}} \tag{5}$$

and the mapping $\mathcal{H}(u)$ is defined by

$$\mathcal{H}(u) = \lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \lambda(\gamma^{\delta\nu} u) \tag{6}$$

for all $u \in \mathcal{A}$.

Proof. Assume $\nu = 1$. Letting (u, v) by (u, u) in (1), we arrive

$$\|2\lambda((\alpha + \beta)u) - 2(\alpha + \beta)\lambda(u)\|_{\mathcal{B}} \le \eta(u, u) \Longrightarrow \|\lambda(\gamma u) - \gamma\lambda(u)\|_{\mathcal{B}} \le \frac{1}{2}\eta(u, u) \tag{7}$$

for all $u \in \mathcal{A}$. It follows from above inequality that

$$\left\| \frac{\lambda(\gamma u)}{\gamma} - \lambda(u) \right\|_{\mathcal{B}} \le \frac{1}{2\gamma}\eta(u, u) \tag{8}$$

for all $u \in \mathcal{A}$. Now replacing u by γu and dividing by γ in (8), we obtain

$$\left\| \frac{\lambda(\gamma^2 u)}{\gamma^2} - \frac{\lambda(\gamma u)}{\gamma} \right\|_{\mathcal{B}} \le \frac{1}{2\gamma^2}\eta(\gamma u, \gamma u) \tag{9}$$

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for all $u \in \mathcal{A}$. From (8) and (9), we get

$$\left\| \frac{\lambda(\gamma^2 u)}{\gamma^2} - \lambda(u) \right\|_{\mathcal{B}} \leq \frac{K}{2\gamma} \left[\eta(u,u) + \frac{\eta(\gamma u, \gamma u)}{\gamma} \right] \tag{10}$$

for all $u \in \mathcal{A}$. Proceeding further and using induction on a positive integer δ, we have

$$\left\| \frac{\lambda(\gamma^\delta u)}{\gamma^\delta} - \lambda(u) \right\|_{\mathcal{B}} \leq \frac{K^{\delta-1}}{2\gamma} \sum_{\chi=0}^{\delta-1} \frac{1}{\gamma^\chi} \eta(\gamma^\chi u, \gamma^\chi u) \tag{11}$$

for all $u \in \mathcal{A}$. The rest of the proof is similar lines to that of Theorem 5. This completes the proof of the Theorem. □

The following corollary is an immediate consequence of Theorem 12 concerning some stabilities of (2).

Corollary 3. *Suppose $\lambda : \mathcal{A} \to \mathcal{B}$ be a mapping and there exists real numbers π and ϖ such that*

$$\left.\begin{array}{r} \|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha+\beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \\ \\ \|\lambda(uv) - \lambda(u) - \lambda(v)\|_{\mathcal{B}} \end{array}\right\} \leq \left\{ \begin{array}{l} \pi, \\ \pi \left\{ ||u||_{\mathcal{A}}^{\varpi} + ||v||_{\mathcal{A}}^{\varpi} \right\}, \\ \pi \; ||u||_{\mathcal{A}}^{\varpi} ||v||_{\mathcal{A}}^{\varpi}, \end{array} \right. \tag{12}$$

for all $u, v \in \mathcal{A}$. Then there exists a unique homomorphism function $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{H}(u)\|_{\mathcal{B}} \leq \left\{ \begin{array}{ll} \dfrac{K^{\delta-1}\pi}{2|1-\gamma|}, & \\ \dfrac{K^{\delta-1}\pi ||u||_{\mathcal{A}}^{\varpi}}{2|\gamma - \gamma^{\varpi}|}, & \varpi \neq 1 \\ \dfrac{K^{\delta-1}\pi ||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1 \end{array} \right. \tag{13}$$

for all $u \in \mathcal{A}$.

3.3. **Derivation Stability Result.**

Theorem 13. *If $\lambda : \mathcal{A} \to \mathcal{A}$ and $\eta : \mathcal{A}^2 \to [0, \infty)$ are functions satisfying the double inequalities*

$$\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha+\beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \leq \eta(u,v) \tag{14}$$

$$\|\lambda(uv) - u\lambda(v) - \lambda(u)v\|_{\mathcal{B}} \leq \eta(u,v) \tag{15}$$

and

$$\lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) = 0 = \lim_{\delta\to\infty} \frac{1}{\gamma^{2\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) \tag{16}$$

for all $u, v \in \mathcal{A}$ where

$$\nu = \pm 1 \qquad \text{and} \qquad \gamma = \alpha + \beta. \tag{17}$$

Then there exists a unique derivation function $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{D}(u)\|_{\mathcal{B}} \leq \frac{K^{\delta-1}}{2\gamma} \sum_{\chi=\frac{1-\nu}{2}}^{\infty} \frac{\eta(\gamma^{\chi\nu} u, \gamma^{\chi\nu} u)}{\gamma^{\chi\nu}} \tag{18}$$

and the mapping $\mathcal{D}(u)$ is defined by

$$\mathcal{D}(u) = \lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \lambda(\gamma^{\delta\nu} u) \tag{19}$$

for all $u \in \mathcal{A}$.

Proof. The proof is similar lines to that of Theorem 6. □

The following corollary is an immediate consequence of Theorem 13 concerning some stabilities of (2).

Corollary 4. *Suppose $\lambda : \mathcal{A} \to \mathcal{A}$ be a mapping and there exists real numbers π and ϖ such that*

$$\left.\begin{array}{r} \|\lambda(\alpha u+\beta v)+\lambda(\beta u+\alpha v)-(\alpha+\beta)(\lambda(u)+\lambda(v))\|_{\mathcal{B}} \\ \|\lambda(uv)-u\lambda(v)-\lambda(u)v\|_{\mathcal{B}} \end{array}\right\} \leq \left\{\begin{array}{l} \pi, \\ \pi\left\{||u||_{\mathcal{A}}^{\varpi}+||v||_{\mathcal{A}}^{\varpi}\right\}, \\ \pi\ ||u||_{\mathcal{A}}^{\varpi}||v||_{\mathcal{A}}^{\varpi}, \end{array}\right. \tag{20}$$

for all $u, v \in \mathcal{A}$. Then there exists a unique derivation function $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying the functional equation (2) and

$$\|\lambda(u)-\mathcal{D}(u)\|_{\mathcal{B}} \leq \left\{\begin{array}{ll} \dfrac{K^{\delta-1}\pi}{2|1-\gamma|}, & \\ \dfrac{K^{\delta-1}\pi||u||_{\mathcal{A}}^{\varpi}}{2|\gamma-\gamma^{\varpi}|}, & \varpi\neq 1 \\ \dfrac{K^{\delta-1}\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma-\gamma^{2\varpi}|}, & 2\varpi\neq 1 \end{array}\right. \tag{21}$$

for all $u \in \mathcal{A}$.

4. Stability Results in C^* Algebras

4.1. C^*-Algebra Definitions.

Definition 14. A Banach algebra A is said to be a C^*- algebra if it satisfies the involution condition

$$f(x^*) = f(x)^*$$

for all $u \in A$.

Definition 15. Let A and B be C^*- algebras. A mapping $H : A \to B$ is called a C^*-algebra homomorphism if

$$H(xy) = H(x)H(y)$$

for all $u, y \in A$.

Definition 16. Let A and B be C^*- algebras. A mapping $D : A \to A$ is called a C^*-algebra derivation if

$$D(xy) = D(x)y + xD(y)$$

for all $u, y \in A$.

In order to establish the stability results, throughout this section let us assume $\mathcal{A}$ is a C^*- algebra with norm $||\cdot||_A$ and $\mathcal{B}$ is a C^*- algebra with norm $||\cdot||_B$.

4.2. **Homomorphism Stability Result.**

Theorem 17. *If $\lambda : \mathcal{A} \to \mathcal{B}$ and $\eta : \mathcal{A}^2 \to [0, \infty)$ are functions satisfying the triple inequalities*

$$\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \leq \eta(u, v) \tag{1}$$

$$\|\lambda(uv) - \lambda(u) - \lambda(v)\|_{\mathcal{B}} \leq \eta(u, v) \tag{2}$$

$$\|\lambda(u^*) - \lambda(u)^*\|_{\mathcal{B}} \leq \eta(u) \tag{3}$$

and

$$\lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) = 0 = \lim_{\delta\to\infty} \frac{1}{\gamma^{2\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) = \lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \eta(\gamma^{\delta\nu} u) \tag{4}$$

for all $u, v \in \mathcal{A}$ where

$$\nu = \pm 1 \qquad \text{and} \qquad \gamma = \alpha + \beta. \tag{5}$$

Then there exists a unique C^*- algebra homomorphism function $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{H}(u)\|_{\mathcal{B}} \leq \frac{1}{2\gamma} \sum_{\chi=\frac{1-\nu}{2}}^{\infty} \frac{\eta(\gamma^{\chi\nu} u, \gamma^{\chi\nu} u)}{\gamma^{\chi\nu}} \tag{6}$$

and the mapping $\mathcal{H}(u)$ is defined by

$$\mathcal{H}(u) = \lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \lambda(\gamma^{\delta\nu} u) \tag{7}$$

for all $u \in \mathcal{A}$.

Proof. By the proof of Theorem 5, there exists a unique homomorphism mapping $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ satisfying (6). From (3) and definition of $\mathcal{H}$, we achieve

$$\begin{aligned}\|\mathcal{H}(u^*) - \mathcal{H}(u)^*\|_{\mathcal{B}} &= \frac{1}{\gamma^{\delta}} \left\|\lambda(\gamma^{\delta} u^*) - \lambda(\gamma^{\delta} u)^*\right\|_{\mathcal{B}} \\ &\leq \frac{1}{\gamma^{\delta}} \eta(\gamma^{\delta} u) \to 0 \quad \text{as } \delta \to \infty.\end{aligned}$$

Therefore

$$\mathcal{H}(u^*) = \mathcal{H}(u)^*$$

for all $u \in \mathcal{A}$. Thus, $\mathcal{H}$ is a C^*-algebra homomorphism. □

The following corollary is an immediate consequence of Theorem 17 concerning some stabilities of (2).

Corollary 5. *Suppose $\lambda : \mathcal{A} \to \mathcal{B}$ be a mapping and there exists real numbers π and ϖ such that*

$$\left.\begin{aligned}\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \\ \|\lambda(uv) - \lambda(u) - \lambda(v)\|_{\mathcal{B}}\end{aligned}\right\} \leq \begin{cases} \pi, \\ \pi \left\{||u||_{\mathcal{A}}^{\varpi} + ||v||_{\mathcal{A}}^{\varpi}\right\}, \\ \pi\ ||u||_{\mathcal{A}}^{\varpi} ||v||_{\mathcal{A}}^{\varpi}, \end{cases} \tag{8}$$

and

$$\|\lambda(u^*) - \lambda(u)^*\|_{\mathcal{B}} \leq \pi ||u||_{\mathcal{A}}^{\varpi} \tag{9}$$

for all $u, v \in \mathcal{A}$. Then there exists a unique C^*- algebra homomorphism function $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{H}(u)\|_{\mathcal{B}} \leq \begin{cases} \dfrac{\pi}{2|1-\gamma|}, & \\ \dfrac{\pi||u||_{\mathcal{A}}^{\varpi}}{2|\gamma - \gamma^{\varpi}|}, & \varpi \neq 1 \\ \dfrac{\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1 \end{cases} \tag{10}$$

for all $u \in \mathcal{A}$.

4.3. **Derivation Stability Result.**

Theorem 18. *If $\lambda : \mathcal{A} \to \mathcal{A}$ and $\eta : \mathcal{A}^2 \to [0, \infty)$ are functions satisfying the triple inequalities*

$$\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \leq \eta(u, v) \tag{11}$$

$$\|\lambda(uv) - u\lambda(v) - \lambda(u)v\|_{\mathcal{B}} \leq \eta(u, v) \tag{12}$$

$$\|\lambda(u^*) - \lambda(u)^*\|_{\mathcal{B}} \leq \eta(u) \tag{13}$$

and

$$\lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \eta(\gamma^{\delta\nu}u, \gamma^{\delta\nu}v) = 0 = \lim_{\delta\to\infty} \frac{1}{\gamma^{2\delta\nu}} \eta(\gamma^{\delta\nu}u, \gamma^{\delta\nu}v) \tag{14}$$

for all $u, v \in \mathcal{A}$ where

$$\nu = \pm 1 \qquad \text{and} \qquad \gamma = \alpha + \beta. \tag{15}$$

Then there exists a unique C^*- algebra derivation function $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{D}(u)\|_{\mathcal{B}} \leq \frac{1}{2\gamma} \sum_{\chi=\frac{1-\nu}{2}}^{\infty} \frac{\eta(\gamma^{\chi\nu}u, \gamma^{\chi\nu}u)}{\gamma^{\chi\nu}} \tag{16}$$

and the mapping $\mathcal{D}(u)$ is defined by

$$\mathcal{D}(u) = \lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \lambda(\gamma^{\delta\nu}u) \tag{17}$$

for all $u \in \mathcal{A}$.

Proof. The proof is similar lines to that of Theorem 6. □

The following corollary is an immediate consequence of Theorem 18 concerning some stabilities of (2).

Corollary 6. *Suppose $\lambda : \mathcal{A} \to \mathcal{A}$ be a mapping and there exists real numbers π and ϖ such that*

$$\left. \begin{array}{r} \|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \\ \|\lambda(uv) - u\lambda(v) - \lambda(u)v\|_{\mathcal{B}} \end{array} \right\} \leq \begin{cases} \pi, \\ \pi\left\{||u||_{\mathcal{A}}^{\varpi} + ||v||_{\mathcal{A}}^{\varpi}\right\}, \\ \pi\ ||u||_{\mathcal{A}}^{\varpi}||v||_{\mathcal{A}}^{\varpi}, \end{cases} \tag{18}$$

and

$$\|\lambda(u^*) - \lambda(u)^*\|_{\mathcal{B}} \leq \pi||u||_{\mathcal{A}}^{\varpi} \tag{19}$$

for all $u, v \in \mathcal{A}$. Then there exists a unique C^*- algebra derivation function $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{D}(u)\|_{\mathcal{B}} \leq \begin{cases} \dfrac{\pi}{2|1-\gamma|}, \\ \dfrac{\pi||u||_{\mathcal{A}}^{\varpi}}{2|\gamma - \gamma^{\varpi}|}, & \varpi \neq 1 \\ \dfrac{\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1 \end{cases} \tag{20}$$

for all $u \in \mathcal{A}$.

5. Stability Results in Lie C^* Algebras

5.1. Lie C^*-Algebra Definitions.

Definition 19. A C^*- algebra A endowed with the Lie product

$$[x, y] = \frac{(xy - yx)}{2}$$

on A, is called a Lie C^*-algebra for all $u, y \in A$.

Definition 20. Let A and B be C^*- algebras. A mapping $H : A \to B$ is called a Lie Lie C^*- algebra homomorphism if

$$H([xy]) = [H(x), H(y)]$$

for all $u, y \in A$.

Definition 21. Let A and B be C^*- algebras. A mapping $D : A \to A$ is called a Lie C^*- derivation if

$$D([xy]) = [D(x), y] + [x, D(y)]$$

for all $u, y \in A$.

In order to establish the stability results, throughout this section let us assume $\mathcal{A}$ is a Lie C^*- algebra with norm $|| \cdot ||_A$ and $\mathcal{B}$ is a Lie C^*- algebra with norm $|| \cdot ||_B$.

5.2. Homomorphism Stability Result.

Theorem 22. *If $\lambda : \mathcal{A} \to \mathcal{B}$ and $\eta : \mathcal{A}^2 \to [0, \infty)$ are functions satisfying the double inequalities*

$$\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \leq \eta(u, v) \tag{1}$$

$$\|\lambda([uv]) - [\lambda(u), \lambda(v)]\|_{\mathcal{B}} \leq \eta(u, v) \tag{2}$$

and

$$\lim_{\delta \to \infty} \frac{1}{\gamma^{\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) = 0 = \lim_{\delta \to \infty} \frac{1}{\gamma^{2\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) \tag{3}$$

for all $u, v \in \mathcal{A}$ where

$$\nu = \pm 1 \qquad and \qquad \gamma = \alpha + \beta. \tag{4}$$

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Then there exists a unique Lie C^*- homomorphism function $\mathcal{H}:\mathcal{A}\to\mathcal{B}$ satisfying the functional equation (2) and

$$\|\lambda(u)-\mathcal{H}(u)\|_{\mathcal{B}}\le\frac{1}{2\gamma}\sum_{\chi=\frac{1-\nu}{2}}^{\infty}\frac{\eta(\gamma^{\chi\nu}u,\gamma^{\chi\nu}u)}{\gamma^{\chi\nu}} \tag{5}$$

and the mapping $\mathcal{H}(u)$ is defined by

$$\mathcal{H}(u)=\lim_{\delta\to\infty}\frac{1}{\gamma^{\delta\nu}}\lambda(\gamma^{\delta\nu}u) \tag{6}$$

for all $u\in\mathcal{A}$.

Proof. By the proof of Theorem 5, there exists a unique homomorphism mapping $\mathcal{H}:\mathcal{A}\to\mathcal{B}$ satisfying (5). From (2) and definition of $\mathcal{H}$, we achieve

$$\begin{aligned}\|\mathcal{H}([uv])=[\mathcal{H}(u),\mathcal{H}(v)]\|_{\mathcal{B}}&=\frac{1}{\gamma^{2\delta}}\left\|\lambda([\gamma^{\delta}u\ \gamma^{\delta}v])-[\lambda(\gamma^{\delta}u),\lambda(\gamma^{\delta}v)]\right\|_{\mathcal{B}}\\&\le\frac{1}{\gamma^{2\delta}}\eta(\gamma^{\delta}u,\gamma^{\delta}v)\to0\quad as\ \ \delta\to\infty.\end{aligned}$$

Therefore

$$\mathcal{H}([uv])=[\mathcal{H}(u),\mathcal{H}(v)]$$

for all $u\in\mathcal{A}$. Thus, $\mathcal{H}$ is a Lie C^*- algebra homomorphism. □

The following corollary is an immediate consequence of Theorem 22 concerning some stabilities of (2).

Corollary 7. *Suppose $\lambda:\mathcal{A}\to\mathcal{B}$ be a mapping and there exists real numbers π and ϖ such that*

$$\left.\begin{array}{r}\|\lambda(\alpha u+\beta v)+\lambda(\beta u+\alpha v)-(\alpha+\beta)(\lambda(u)+\lambda(v))\|_{\mathcal{B}}\\ \\ \|\lambda(uv)-\lambda(u)-\lambda(v)\|_{\mathcal{B}}\end{array}\right\}\le\left\{\begin{array}{l}\pi,\\ \pi\left\{||u||_{\mathcal{A}}^{\varpi}+||v||_{\mathcal{A}}^{\varpi}\right\},\\ \pi\ ||u||_{\mathcal{A}}^{\varpi}||v||_{\mathcal{A}}^{\varpi},\end{array}\right. \tag{7}$$

for all $u,v\in\mathcal{A}$. Then there exists a unique Lie C^*- algebra homomorphism function $\mathcal{H}:\mathcal{A}\to\mathcal{B}$ satisfying the functional equation (2) and

$$\|\lambda(u)-\mathcal{H}(u)\|_{\mathcal{B}}\le\left\{\begin{array}{ll}\dfrac{\pi}{2|1-\gamma|}, & \\ \dfrac{\pi||u||_{\mathcal{A}}^{\varpi}}{2|\gamma-\gamma^{\varpi}|}, & \varpi\neq1\\ \dfrac{\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma-\gamma^{2\varpi}|}, & 2\varpi\neq1\end{array}\right. \tag{8}$$

for all $u\in\mathcal{A}$.

5.3. **Derivation Stability Result.**

Theorem 23. *If $\lambda:\mathcal{A}\to\mathcal{A}$ and $\eta:\mathcal{A}^2\to[0,\infty)$ are functions satisfying the double inequalities*

$$\|\lambda(\alpha u+\beta v)+\lambda(\beta u+\alpha v)-(\alpha+\beta)(\lambda(u)+\lambda(v))\|_{\mathcal{B}}\le\eta(u,v) \tag{9}$$

$$\|\lambda([uv])-[\lambda(u)v]-[u,\lambda(v)]\|_{\mathcal{B}}\le\eta(u,v) \tag{10}$$

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and

$$\lim_{\delta\to\infty}\frac{1}{\gamma^{\delta\nu}}\eta(\gamma^{\delta\nu}u,\gamma^{\delta\nu}v)=0=\lim_{\delta\to\infty}\frac{1}{\gamma^{2\delta\nu}}\eta(\gamma^{\delta\nu}u,\gamma^{\delta\nu}v) \tag{11}$$

for all $u,v\in\mathcal{A}$ *where*

$$\nu=\pm1 \qquad and \qquad \gamma=\alpha+\beta. \tag{12}$$

Then there exists a unique Lie C^*- *algebra derivation function* $\mathcal{D}:\mathcal{A}\to\mathcal{A}$ *satisfying the functional equation (2) and*

$$\|\lambda(u)-\mathcal{D}(u)\|_{\mathcal{B}}\le\frac{1}{2\gamma}\sum_{\chi=\frac{1-\nu}{2}}^{\infty}\frac{\eta(\gamma^{\chi\nu}u,\gamma^{\chi\nu}u)}{\gamma^{\chi\nu}} \tag{13}$$

and the mapping $\mathcal{D}(u)$ *is defined by*

$$\mathcal{D}(u)=\lim_{\delta\to\infty}\frac{1}{\gamma^{\delta\nu}}\lambda(\gamma^{\delta\nu}u) \tag{14}$$

for all $u\in\mathcal{A}$.

Proof. By the proof of Theorem 6, there exists a unique additive mapping $\mathcal{D}:\mathcal{A}\to\mathcal{A}$ satisfying (13). Also, the mapping $\mathcal{D}:\mathcal{A}\to\mathcal{A}$ given by

$$\mathcal{D}(u)=\lim_{\delta\to\infty}\frac{1}{\gamma^{\delta}}\lambda(\gamma^{\delta}u)$$

for all $u\in\mathcal{A}$.

From (10) and by definition of $\mathcal{D}$, we achieve

$$\begin{aligned}\|\mathcal{D}([uv])-[\mathcal{D}(u),v]-[u,\mathcal{D}(v)]\|_{\mathcal{B}}&=\frac{1}{\gamma^{2\delta}}\left\|\lambda([\gamma^{\delta}u\ \gamma^{\delta}v])-[\lambda(\gamma^{\delta}u),\gamma^{\delta}v]-[\gamma^{\delta}u,\lambda(\gamma^{\delta}v)]\right\|_{\mathcal{B}}\\&\le\frac{1}{\gamma^{2\delta}}\eta(\gamma^{\delta}u,\gamma^{\delta}v)\to0\quad as\ \ \delta\to\infty.\end{aligned}$$

Therefore

$$\mathcal{D}([uv])=[\mathcal{D}(u),v]+[u,\mathcal{D}(v)]$$

for all $u,v\in\mathcal{A}$. Thus, $\mathcal{D}$ is a Lie C^*- algebra derivation. □

The following corollary is an immediate consequence of Theorem 23 concerning some stabilities of (2).

Corollary 8. *Suppose* $\lambda:\mathcal{A}\to\mathcal{A}$ *be a mapping and there exists real numbers* π *and* ϖ *such that*

$$\left.\begin{array}{r}\|\lambda(\alpha u+\beta v)+\lambda(\beta u+\alpha v)-(\alpha+\beta)(\lambda(u)+\lambda(v))\|_{\mathcal{B}}\\ \\ \|\lambda([uv])-[\lambda(u)v]-[u,\lambda(v)]\|_{\mathcal{B}}\end{array}\right\}\le\left\{\begin{array}{l}\pi,\\ \pi\left\{||u||_{\mathcal{A}}^{\varpi}+||v||_{\mathcal{A}}^{\varpi}\right\},\\ \pi\ ||u||_{\mathcal{A}}^{\varpi}||v||_{\mathcal{A}}^{\varpi},\end{array}\right. \tag{15}$$

for all $u,v\in\mathcal{A}$. *Then there exists a unique Lie* C^*- *algebra derivation function* $\mathcal{D}:\mathcal{A}\to\mathcal{A}$ *satisfying the functional equation (2) and*

$$\|\lambda(u)-\mathcal{D}(u)\|_{\mathcal{B}}\le\begin{cases}\dfrac{\pi}{2|1-\gamma|},\\ \dfrac{\pi||u||_{\mathcal{A}}^{\varpi}}{2|\gamma-\gamma^{\varpi}|}, & \varpi\ne1\\ \dfrac{\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma-\gamma^{2\varpi}|}, & 2\varpi\ne1\end{cases} \tag{16}$$

for all $u \in \mathcal{A}$.

6. Stability Results in Jordan C^* Algebras

6.1. JC^*-Algebra Definitions.

Definition 24. A Jordan C^*- algebra A endowed with a anticommutator product

$$u \circ y = \frac{(xy + yx)}{2}$$

on A, is called a JC^*-algebra for all $u, y \in A$.

Definition 25. Let A and B be C^*- algebras. A mapping $H : A \to B$ is called a JC^*-algebra homomorphism if

$$H(x \circ y) = H(x) \circ H(y)$$

for all $u, y \in A$.

Definition 26. Let A and B be C^*- algebras. A mapping $D : A \to A$ is called a JC^*-algebra derivation if

$$D(x \circ y) = D(x) \circ y + x \circ D(y)$$

for all $u, y \in A$.

In order to establish the stability results, throughout this section let us assume $\mathcal{A}$ is a JC^*- algebra with norm $|| \cdot ||_A$ and $\mathcal{B}$ is a JC^*- algebra with norm $|| \cdot ||_B$.

6.2. Homomorphism Stability Result.

Theorem 27. *If* $\lambda : \mathcal{A} \to \mathcal{B}$ *and* $\eta : \mathcal{A}^2 \to [0, \infty)$ *are functions satisfying the double inequalities*

$$\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \leq \eta(u, v) \quad (1)$$

$$\|\lambda(u \circ v) - \lambda(u) \circ \lambda(v)\|_{\mathcal{B}} \leq \eta(u, v) \quad (2)$$

and

$$\lim_{\delta \to \infty} \frac{1}{\gamma^{\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) = 0 = \lim_{\delta \to \infty} \frac{1}{\gamma^{2\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) \quad (3)$$

for all $u, v \in \mathcal{A}$ *where*

$$\nu = \pm 1 \qquad and \qquad \gamma = \alpha + \beta. \quad (4)$$

Then there exists a unique JC^*- *homomorphism function* $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ *satisfying the functional equation (2) and*

$$\|\lambda(u) - \mathcal{H}(u)\|_{\mathcal{B}} \leq \frac{1}{2\gamma} \sum_{\chi = \frac{1-\nu}{2}}^{\infty} \frac{\eta(\gamma^{\chi\nu} u, \gamma^{\chi\nu} u)}{\gamma^{\chi\nu}} \quad (5)$$

and the mapping $\mathcal{H}(u)$ *is defined by*

$$\mathcal{H}(u) = \lim_{\delta \to \infty} \frac{1}{\gamma^{\delta\nu}} \lambda(\gamma^{\delta\nu} u) \quad (6)$$

for all $u \in \mathcal{A}$.

Proof. By the proof of Theorem 5, there exists a unique homomorphism mapping $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ satisfying (5). From (2) and definition of $\mathcal{H}$, we achieve

$$\begin{aligned}\|\mathcal{H}(u \circ v) - \mathcal{H}(u) \circ \mathcal{H}(v)\|_{\mathcal{B}} &= \frac{1}{\gamma^{2\delta}} \left\|\lambda(\gamma^{\delta} u \circ \gamma^{\delta} v) - \lambda(\gamma^{\delta} u) \circ \lambda(\gamma^{\delta} v)\right\|_{\mathcal{B}} \\ &\leq \frac{1}{\gamma^{2\delta}} \eta(\gamma^{\delta} u, \gamma^{\delta} v) \to 0 \quad as \ \delta \to \infty.\end{aligned}$$

Therefore

$$\mathcal{H}(u \circ v) = \mathcal{H}(u) \circ \mathcal{H}(v)$$

for all $u \in \mathcal{A}$. Thus, $\mathcal{H}$ is a JC^*- algebra homomorphism. □

The following corollary is an immediate consequence of Theorem 27 concerning some stabilities of (2).

Corollary 9. *Suppose $\lambda : \mathcal{A} \to \mathcal{B}$ be a mapping and there exists real numbers π and ϖ such that*

$$\left.\begin{array}{r}\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \\ \\ \|\lambda(u \circ v) - \lambda(u) \circ \lambda(v)\|_{\mathcal{B}}\end{array}\right\} \leq \begin{cases}\pi, \\ \pi \left\{||u||_{\mathcal{A}}^{\varpi} + ||v||_{\mathcal{A}}^{\varpi}\right\}, \\ \pi \ ||u||_{\mathcal{A}}^{\varpi}||v||_{\mathcal{A}}^{\varpi},\end{cases} \tag{7}$$

for all $u, v \in \mathcal{A}$. Then there exists a unique JC^*- algebra homomorphism function $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{H}(u)\|_{\mathcal{B}} \leq \begin{cases}\dfrac{\pi}{2|1-\gamma|}, \\ \dfrac{\pi||u||_{\mathcal{A}}^{\varpi}}{2|\gamma - \gamma^{\varpi}|}, & \varpi \neq 1 \\ \dfrac{\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1\end{cases} \tag{8}$$

for all $u \in \mathcal{A}$.

6.3. **Derivation Stability Result.**

Theorem 28. *If $\lambda : \mathcal{A} \to \mathcal{A}$ and $\eta : \mathcal{A}^2 \to [0, \infty)$ are functions satisfying the double inequalities*

$$\|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \leq \eta(u, v) \tag{9}$$

$$\|\lambda(u \circ v) - \lambda(u) \circ v - u \circ \lambda(v)\|_{\mathcal{B}} \leq \eta(u, v) \tag{10}$$

and

$$\lim_{\delta \to \infty} \frac{1}{\gamma^{\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) = 0 = \lim_{\delta \to \infty} \frac{1}{\gamma^{2\delta\nu}} \eta(\gamma^{\delta\nu} u, \gamma^{\delta\nu} v) \tag{11}$$

for all $u, v \in \mathcal{A}$ where

$$\nu = \pm 1 \qquad and \qquad \gamma = \alpha + \beta. \tag{12}$$

Then there exists a unique JC^*- algebra derivation function $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{D}(u)\|_{\mathcal{B}} \leq \frac{1}{2\gamma} \sum_{\chi = \frac{1-\nu}{2}}^{\infty} \frac{\eta(\gamma^{\chi\nu} u, \gamma^{\chi\nu} u)}{\gamma^{\chi\nu}} \tag{13}$$

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and the mapping $\mathcal{D}(u)$ is defined by

$$\mathcal{D}(u) = \lim_{\delta\to\infty} \frac{1}{\gamma^{\delta\nu}} \lambda(\gamma^{\delta\nu} u) \tag{14}$$

for all $u \in \mathcal{A}$.

Proof. By the proof of Theorem 6, there exists a unique additive mapping $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying (13). Also, the mapping $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ given by

$$\mathcal{D}(u) = \lim_{\delta\to\infty} \frac{1}{\gamma^{\delta}} \lambda(\gamma^{\delta} u)$$

for all $u \in \mathcal{A}$.

From (10) and by definition of $\mathcal{D}$, we achieve

$$\begin{aligned} \|\mathcal{D}(u \circ v) - \mathcal{D}(u) \circ v - u \circ \mathcal{D}(v)]\|_{\mathcal{B}} &= \frac{1}{\gamma^{2\delta}} \left\|\lambda(\gamma^{\delta} u \circ \gamma^{\delta} v]) - \lambda(\gamma^{\delta} u) \circ \gamma^{\delta} v - \gamma^{\delta} u \circ \lambda(\gamma^{\delta} v)\right\|_{\mathcal{B}} \\ &\leq \frac{1}{\gamma^{2\delta}} \eta(\gamma^{\delta} u, \gamma^{\delta} v) \to 0 \quad as \;\; \delta \to \infty. \end{aligned}$$

Therefore

$$\mathcal{D}(u \circ v) = \mathcal{D}(u) \circ v + u \circ \mathcal{D}(v)$$

for all $u, v \in \mathcal{A}$. Thus, $\mathcal{D}$ is a JC^*- algebra derivation. □

The following corollary is an immediate consequence of Theorem 28 concerning some stabilities of (2).

Corollary 10. *Suppose $\lambda : \mathcal{A} \to \mathcal{A}$ be a mapping and there exists real numbers π and ϖ such that*

$$\left.\begin{array}{l} \|\lambda(\alpha u + \beta v) + \lambda(\beta u + \alpha v) - (\alpha + \beta)(\lambda(u) + \lambda(v))\|_{\mathcal{B}} \\ \qquad\qquad \|\lambda(u \circ v) - \lambda(u) \circ v - u \circ \lambda(v)\|_{\mathcal{B}} \end{array}\right\} \leq \left\{ \begin{array}{l} \pi, \\ \pi\left\{||u||_{\mathcal{A}}^{\varpi} + ||v||_{\mathcal{A}}^{\varpi}\right\}, \\ \pi\; ||u||_{\mathcal{A}}^{\varpi} ||v||_{\mathcal{A}}^{\varpi}, \end{array}\right. \tag{15}$$

for all $u, v \in \mathcal{A}$. Then there exists a unique JC^*- algebra derivation function $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying the functional equation (2) and

$$\|\lambda(u) - \mathcal{D}(u)\|_{\mathcal{B}} \leq \left\{ \begin{array}{ll} \dfrac{\pi}{2|1-\gamma|}, & \\ \dfrac{\pi||u||_{\mathcal{A}}^{\varpi}}{2|\gamma - \gamma^{\varpi}|}, & \varpi \neq 1 \\ \dfrac{\pi||u||_{\mathcal{A}}^{2\varpi}}{2|\gamma - \gamma^{2\varpi}|}, & 2\varpi \neq 1 \end{array}\right. \tag{16}$$

for all $u \in \mathcal{A}$.

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