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HYERS-ULAM STABILITY OF A SPECIAL TYPE LINEAR DIFFERENTIAL EQUATION OF FIRST ORDER USING TAYLOR'S SERIES

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Abstract. In this paper, we prove the approximate solution of the special type first order linear differential equation by applying initial and boundary conditions. That is, we prove the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the special type linear differential equations of first order with initial and boundary conditions using Taylor's series formula.

1. Introduction

The theory of stability is an important branch of the qualitative theory of differential equations. In 1940, Ulam [1] posed a problem concerning the stability of functional equation: "Give conditions in order for a linear function near an approximately linear function to exist." A year later, Hyers [2] gave an answer to the problem of Ulam for Cauchy additive functional equation defined on Banach spaces. Thereafter, Aoki [3], Bourgin [4] and Rassias [5] improved the result reported in [2]. After that, many mathematicians have extended the Ulam's problem to other functional equations on various spaces in different directions [6, 7, 8, 14, 15, 16, 17].

Definitions of both Hyers-Ulam stability and Hyers-Ulam-Rassias stability have applicable significance since it means that if one is studying an Hyers-Ulam stable or Hyers-Ulam-Rassias stable system then one does not have to reach the exact solution. (Which is usually is quite difficult or time consuming). This is quite useful in many applications. For example, numerical analysis, optimization, biology, economics, dynamic programming, wireless sensor networks, physics, chemistry, geometry and etc., where finding the exact solution is quite difficult.

A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation

$$\phi\left(f, x, x^{'}, x^{''}, ...x^{(n)}\right) = 0$$

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has the Hyers - Ulam stability if for a given $\epsilon > 0$ and a function x such that

$$\left|\phi\left(f, x, x^{'}, x^{''}, ...x^{(n)}\right)\right| \leq \epsilon,$$

there exists a solution x_a of the differential equation such that

$$|x(t) - x_a(t)| \leq K(\epsilon)$$

and $\lim\limits_{\epsilon \to 0} K(\epsilon) = 0$.

Oblaza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [18, 19]). Thereafter, Alsina and Ger published their paper [20], which handles the Hyers-Ulam stability of the linear differential equation $y'(t) = y(t)$. They proved in [20] the following theorem.

Theorem 1. *Assume that a differentiable function $f : I \longrightarrow R$ is a solution of the differential inequality*

$$\|x'(t) - x(t)\| \leq \epsilon.$$

where I is an open sub interval of R . Then there exists a solution $g : I \longrightarrow R$ of the differential equation $x^{'}(t) = x(t)$ such that for any $t \in I$, we have,

$$\|f(t) - g(t)\| \leq \ 3\epsilon.$$

This result of C. Alsina and R. Ger [20] has been generalized by Takahasi [21]. They proved in [21] that the Hyers-Ulam stability holds true for the Banach Space valued differential equation $x'(t) = \lambda x(t)$.

Indeed, the Hyers-Ulam stability has been proved for the first order linear differential equations in more general settings [8, 9, 11, 12].

Using the approach as in [1], Miura, Takahasi and Choda [12], Takahasi, Miura and Miyajima [21] proved that the Hyers-Ulam stability holds true for the differential equation

$$x' = \lambda\ x,$$

while Jung [10] proved a similar result for the differential equation $\phi(t)x'(t) = x$.

I. A. Rus [13] discussed four types of Ulam stability: Ulam-Hyers stability, Generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and Generalized Ulam-Hyers-Rassias stability of the Ordinary Differential Equation

$$u'(t) = A(u(t)) + f(t, u(t)), t \in [a, b].$$

Those previous results were extended to the Hyers-Ulam stability of linear differential equations of first order, second order, third order and higher orders in [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36].

In this paper, we study the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the special type first order linear differential equation of the form

$$x'(t) + (p(t) - \alpha(t))\, x(t) = 0 \tag{1}$$

where $x \in C(I)$, $p(t) \in C(I)$, and $\alpha(t)$ is a bounded for all sufficiently large t in $\mathbb{R}$ with initial condition $x(a) = 0$ and with boundary conditions

$$x(a) = x(b) = 0 \tag{2}$$

whereas $I = [a, b]$, $-\infty < a < b < \infty$ using Taylor's series.

2. Preliminaries

First of all, we give the definition of the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of a differential equation (1) with initial and boundary conditions.

Definition 2. We say that the differential equation (1) has the Hyers-Ulam stability with boundary conditions (2), if there exists a positive constant K satisfies the following properties: For every $\epsilon > 0$ and $x \in C([a, b])$ satisfying the inequality

$$|x'(t) + (p(t) - \alpha(t))\, x(t)| \le \epsilon,$$

with $x(a) = x(b) = 0$, then there exists some $y \in C([a, b])$ satisfying

$$y'(t) + (p(t) - \alpha(t))\, y(t) = 0$$

with $y(a) = y(b) = 0$, such that $|x(t) - y(t)| \le K\epsilon$.

Definition 3. We say that the differential equation (1) has the Hyers-Ulam stability with initial conditions, if there exists a positive constant K satisfies the following properties: For every $\epsilon > 0$ and $x \in C([a, b])$ satisfying the inequality

$$|x'(t) + (p(t) - \alpha(t))\, x(t)| \le \epsilon,$$

with $x(a) = 0$, then there exists some $y \in C([a, b])$ satisfying

$$y'(t) + (p(t) - \alpha(t))\, y(t) = 0$$

with $y(a) = 0$, such that $|x(t) - y(t)| \le K\epsilon$.

If the above Definitions is also true when we replace ϵ with $\phi(t)\ \epsilon$, where $\phi : I \to [0, \infty)$ are functions not depending on $x(t)$ and $y(t)$ explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability (or the Hyers-Ulam-Rassias stability).

Definition 4. We say that the differential equation (1) has the Hyers-Ulam-Rassias stability with $\phi(t)$, where $\phi : \mathbb{R} \to [0, \infty)$ and boundary conditions (2), if there exists a positive constant K such that the following conditions are holds: For every $\epsilon > 0$, and $x \in C([a, b])$, if

$$|x'(t) + (p(t) - \alpha(t))\, x(t)| \le \phi(t)\epsilon,$$

and $x(a) = x(b) = 0$, then there exists some $y \in C([a, b])$ satisfying

$$y'(t) + (p(t) - \alpha(t))\, y(t) = 0$$

and $y(a) = y(b) = 0$, such that $|x(t) - y(t)| \le K\phi(t)\epsilon$.

Definition 5. We say that the differential equation (1) has the Hyers-Ulam-Rassias stability with initial condition and $\phi(t)$, where $\phi : \mathbb{R} \to [0, \infty)$, if there exists a positive constant K satisfies the following conditions: For every $\epsilon > 0$, and $x \in C([a, b])$, if

$$|x'(t) + (p(t) - \alpha(t))\, x(t)| \ \leq \phi(t)\epsilon,$$

and $x(a) = 0$, then there exists some $y \in C([a, b])$ satisfying

$$y'(t) + (p(t) - \alpha(t))\, y(t) = 0$$

and $y(a) = 0$, such that $|x(t) - y(t)| \leq K\phi(t)\epsilon$.

We call such K as a Hyers-Ulam stability and Hyers-Ulam-Rassias stability constants for the differential equation respectively.

3. Ulam Stability of Differential Equations using Boundary Conditions

Now, we are going to prove the Hyers-Ulam stability for the linear differential equation (1) with boundary conditions (2).

Theorem 6. *Let* $\max |p(t) - \alpha(t)| < \dfrac{2}{(b-a)}$ *for* $t \in [a, b]$*. Then, the differential equation* (1) *has the Hyers-Ulam stability with boundary conditions* (2)*.*

Proof. For every $\epsilon > 0$, there exists $x \in C([a, b])$, such that

$$|x'(t) + (p(t) - \alpha(t))\, x(t)| \ \leq \ \epsilon,$$

with $x(a) = x(b) = 0$. Let us define $M = \max \{|x(t)| \ : \ t \in [a, b]\}$. Since $x(a) = x(b) = 0$, there exists $t_0 \in (a, b)$ such that $|x(t_0)| = M$. By Taylor's series formula, we have

$$x(a) = x(t_0) + x'(\xi)(t_0 - a) \tag{3}$$

$$x(b) = x(t_0) + x'(\xi)(b - t_0) \tag{4}$$

We have $x(a) = 0$, and so equation (3) becomes

$$x(t_0) + x'(\xi)(t_0 - a) = 0.$$

Thus, we have $|x'(\xi)| = \dfrac{M}{(t_0 - a)}$. Similarly, from $x(b) = 0$ the relation (4) can be converted to

$$x(t_0) + x'(\xi)(b - t_0) = 0.$$

So, we have $|x'(\xi)| = \dfrac{M}{(b - t_0)}$. On the other hand, for $t_0 \in (a, \frac{a+b}{2}]$, we obtain

$$\frac{M}{(t_0 - a)} \geq \frac{M}{\frac{(b-a)}{2}} = \frac{2M}{(b-a)}. \tag{5}$$

Now, if $t_0 \in [\frac{a+b}{2}, b)$, then

$$\frac{M}{(t_0 - b)} \geq \frac{M}{\frac{(b-a)}{2}} = \frac{2M}{(b-a)}. \tag{6}$$

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Using (5) and (6), we have $\max|x(t)| \leq \frac{(b-a)}{2} \max|x'(t)|$. Hence,

$$\max|x(t)| \leq \frac{(b-a)}{2}\{\max|x'(t)+(p(t)-\alpha(t))\,x(t)-(p(t)-\alpha(t))\,x(t)|\}$$
$$\leq \frac{(b-a)}{2}\{\max|x'(t)+(p(t)-\alpha(t))\,x(t)|+\max|(p(t)-\alpha(t))|\ \max|x(t)|\}.$$

Now, let us choose $\rho = \frac{(b-a)}{2} \max|(p(t)-\alpha(t))|$. Then, we obtain that

$$\max|x(t)| \leq \frac{(b-a)}{2}\epsilon + \rho\ \max|x(t)| \ \Rightarrow\ \max|x(t)| \leq \frac{(b-a)}{2\,(1-\rho)}\ \epsilon.$$

Let us consider $K = \frac{(b-a)}{2\,(1-\rho)}$. So, we have $\max|x(t)| \leq K\epsilon$. Obviously, $y_0(t) \equiv 0$ is a solution of the differential equation $x'(t)-(p(t)-\alpha(t))\,x(t)=0$ with boundary conditions $x(a)=x(b)=0$. Therefore,

$$|x(t)-y_0(t)| \leq\ K\epsilon.$$

Hence by the virtue of Definition 2 the differential equation (1) has the Hyers-Ulam stability with boundary conditions (2). □

The following corollaries shows that the Hyers-Ulam-Rassias stability of the first order linear differential equation (1) with boundary conditions (2). Use the same approach of Theorem 6, we can easily prove the following corollary.

When we replace ϵ by $\phi(t)\epsilon$ and $K\epsilon$ by $K\phi(t)\epsilon$ in Theorem 6, we arrive the result. But for the sake of completion, we include some part of the proof.

Corollary 1. *If* $\max|p(t)-\alpha(t)| < \frac{2}{(b-a)}$ *for* $t \in [a,b]$. *For every* $\epsilon > 0$, *there exists a positive constant* K *such that* $x \in C([a,b])$ *satisfying the inequality*

$$|x'(t)+(p(t)-\alpha(t))\,x(t)| \leq \phi(t)\epsilon,$$

with boundary conditions $x(a)=x(b)=0$, *then there exists some* $y \in C([a,b])$ *satisfies the differential equations*

$$y'(t)+(p(t)-\alpha(t))\,y(t)=0$$

with $y(a)=y(b)=0$, *such that* $|x(t)-y(t)| \leq K\phi(t)\epsilon$.

Proof. Given that, for every $\epsilon > 0$, there exists $x \in C([a,b])$, such that

$$|x'(t)+(p(t)-\alpha(t))\,x(t)|\ \leq\ \epsilon\phi(t),$$

with $x(a)=x(b)=0$. Let us define $M = \max\{|x(t)|\ :\ t \in [a,b]\}$. Since $x(a)=x(b)=0$, there exists $t_0 \in (a,b)$ such that $|x(t_0)| = M$. By Taylor's series formula, we have

$$x(a) = x(t_0) + x'(\xi)(t_0 - a) \tag{7}$$

$$x(b) = x(t_0) + x'(\xi)(b - t_0) \tag{8}$$

We have $x(a)=0$, and so equation (3) becomes

$$x(t_0) + x'(\xi)(t_0 - a) = 0.$$

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Thus, we have $|x'(\xi)| = \dfrac{M}{(t_0 - a)}$. Similarly, from $x(b) = 0$ the relation (8) gives that

$$x(t_0) + x'(\xi)(b - t_0) = 0.$$

So, we have $|x'(\xi)| = \dfrac{M}{(b - t_0)}$. On the other hand, for $t_0 \in (a, \frac{a+b}{2}]$, we obtain

$$\frac{M}{(t_0 - a)} \geq \frac{M}{\frac{(b-a)}{2}} = \frac{2M}{(b-a)}. \tag{9}$$

Now, if $t_0 \in [\frac{a+b}{2}, b)$, then

$$\frac{M}{(t_0 - b)} \geq \frac{M}{\frac{(b-a)}{2}} = \frac{2M}{(b-a)}. \tag{10}$$

Using (9) and (10), we have $\max |x(t)| \leq \dfrac{(b-a)}{2} \max |x'(t)|$. Hence,

$$\begin{aligned} \max |x(t)| &\leq \frac{(b-a)}{2} \{ \max |x'(t) + (p(t) - \alpha(t))\, x(t) - (p(t) - \alpha(t))\, x(t)| \} \\ &\leq \frac{(b-a)}{2} \{ \max |x'(t) + (p(t) - \alpha(t))\, x(t)| + \max |(p(t) - \alpha(t))| \ \max |x(t)| \}. \end{aligned}$$

Now, let us choose $\rho = \dfrac{(b-a)}{2} \max |(p(t) - \alpha(t))|$. Then, we obtain that

$$\max |x(t)| \leq \frac{(b-a)}{2} \epsilon\phi(t) + \rho \ \max |x(t)| \ \Rightarrow \ \max |x(t)| \leq \frac{(b-a)}{2\,(1-\rho)} \epsilon\phi(t).$$

Consider $K = \dfrac{(b-a)}{2\,(1-\rho)}$. So, we have $\max |x(t)| \leq K\phi(t)\epsilon$. Obviously, $y_0(t) \equiv 0$ is a solution of the differential equation

$$x'(t) - (p(t) - \alpha(t))\, x(t) = 0$$

with boundary conditions $x(a) = x(b) = 0$. Therefore,

$$|x(t) - y_0(t)| \leq \ K\phi(t)\epsilon.$$

Hence by the virtue of Definition 2 the differential equation (1) has the Hyers-Ulam stability with boundary conditions (2).

Then by the virtue of the Definition 4, the first order differential equation (1) has the Hyers-Ulam-Rassias stability with boundary conditions (2). □

Finally, we are going to study the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the linear differential equation (1) with initial conditions.

Theorem 7. *If* $\max |(p(t) - \alpha(t))| < \dfrac{1!}{(b-a)}$ *for* $t \in [a, b]$. *Then the differential equation* (1) *has the Hyers-Ulam stability with initial condition.*

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Proof. For every $\epsilon > 0$, there exists $x \in C([a,b])$, such that

$$|x'(t) + (p(t) - \alpha(t))\, x(t)| \leq \epsilon.$$

By Taylor's formula, we arrive at

$$x(t) = x(a) + \frac{x'(\zeta)}{1!}(t-a) \tag{11}$$

Using the initial condition $x(a) = 0$, then (11) becomes $x(t) = x'(\zeta)(t-a)$ and thus

$$\max |x(t)| \leq \max |x'(t)|\, (b-a)$$

so, we obtain

$$\begin{aligned} \max |x(t)| &\leq \frac{(b-a)}{1!} \{\max |x'(t) + (p(t) - \alpha(t))\, x(t) - (p(t) - \alpha(t))\, x(t)|\} \\ &\leq \frac{(b-a)}{1!} \{\max |x'(t) + (p(t) - \alpha(t))\, x(t)| + \max |(p(t) - \alpha(t))| \ \max |x(t)|\}. \end{aligned}$$

Let us choose $\eta = \dfrac{(b-a)}{1!} \ \max |(p(t) - \alpha(t))|$. Then

$$\max |x(t)| \leq \frac{(b-a)}{1!}\epsilon + \eta \ \max |x(t)|.$$

Hence, we have $\max |x(t)| \leq K\ \epsilon$, where

$$K \ = \ \frac{(b-a)}{1!\ (1-\eta)}.$$

Hence, $\max |x(t)| \leq K\epsilon$. It is clear that $y_0(t) \equiv 0$ is a solution of the differential equation

$$x'(t) - (p(t) - \alpha(t))\, x(t) = 0$$

with the initial conditions $y(a) = 0$. Thus,

$$|x(t) - y_0(t)| \leq \ K\epsilon.$$

Therefore, by the virtue of Definition 3 the differential equation (1) has the Hyers-Ulam stability with initial conditions. □

The following corollaries shows that the Hyers-Ulam-Rassias stability of the first order linear differential equation (1) with initial conditions. By the similar manner of Theorem 7, we can easily prove the following corollary.

Corollary 2. *If* $\max |p(t) - \alpha(t)| < \dfrac{1!}{(b-a)}$ *for* $t \in [a,b]$*. For every* $\epsilon > 0$*, there exists a positive constant* K *such that* $x \in C([a,b])$ *satisfying the inequality*

$$|x'(t) + (p(t) - \alpha(t))\, x(t)| \leq \phi(t)\epsilon,$$

with initial condition $x(a) = 0$*, then there exists some* $y \in C([a,b])$ *satisfies the differential equations*

$$y'(t) + (p(t) - \alpha(t))\, y(t) = 0$$

with initial condition $y(a) = 0$*, such that* $|x(t) - y(t)| \leq K\phi(t)\epsilon$.

Proof. When we replace ϵ by $\phi(t)\epsilon$ and $K\epsilon$ by $K\phi(t)\epsilon$ in Theorem 7, we arrive the result. □

If the above corollary holds good, then by the virtue of the Definition 5, the first order differential equation (1) has the Hyers-Ulam-Rassias stability with initial condition.

4. Conclusion

In this paper, we proved the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the special type linear differential equations of first order with initial and boundary conditions using Taylor's series formula. This paper provides another method to study Ulam stability for first order differential equations.

References

[1] **S.M. Ulam**, *A Collection of Mathematical Problems*, Interscience Publishers, New York, (1960).

[2] **D.H. Hyers**, On the stability of the linear functional equation, *Proceeding of the National Academy of Sciences of the Unoted States of America*, **27**, (1941), 222-224.

[3] **T. Aoki**, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan*, **2**, (1950), 64-66.

[4] **D.G. Bourgin**, Classes of transformations and bordering transformations, *Bull. Amer. Math. Soc.*, **57**, (1951), 223-237.

[5] **Th.M. Rassias**, On the stability of the linear mapping in Banach Spaces, *Proc. Amer. Math. Soc.*, **72**, (1978), 297-300.

[6] **M. Burger, N. Ozawa, and A. Thom**, On Ulam Stability, *Israel J. Math.*, **193**, (2013), 109-129.

[7] **L. Cadariu, L. Gavruta and P. Gavruta**, Fixed points and generalized Hyers-Ulam stability, *Abstr. Appl. Anal.,* **2012** (2012), Article ID 712743, 10 pages.

[8] **S.M. Jung**, Hyers-Ulam-Rassias stability of Functional equation in nonlinear Analysis, *Springer Optimization and Its Applications*, **Vol. 48**, Springer, New York, 2011.

[9] **S.M. Jung**, *Hyers-Ulam Stability of Linear Differential Equations of First Order (III)*, J. Math. Anal. Appl., **311** (2005) 139-146.

[10] **S.M. Jung**, *Hyers-Ulam Stability of Linear Differential Equations of First Order*, Appl. Math. Lett., **17** (2004) 1135-1140.

[11] **S.M. Jung**, *Hyers-Ulam Stability of a System of First Order Linear Differential Equations with Constant Coefficients*, J. Math. Anal. Appl., **320** (2006) 549-561.

[12] **T. Miura, S. Miyajima, S. E. Takahasi**, *A Characterization of Hyers-Ulam Stability of First Order Linear Differential Operators*, J. Math. Anal. Appl., **286** (2003) 136-146.

[13] **I. A. Rus**, *Ulam Stabilities of Ordinary Differential Equations in a Banach Space*, Carpathian J. Math. **26 (1)** (2010) 103-107.

[14] **M. Almahalebi, A. Chahbi and S. Kabbaj**, A Fixed point approach to the stability of a bi-cubic functional equations in 2-Banach spaces, *Palestine J. Math.*, **vol. 5 (2)**, 2016, 220-227.

[15] **R. Murali, Matina J. Rassias and V. Vithya**, The General Solution and stability of Nonadecic Functional Equation in Matrix Normed Spaces, *Malaya J. Mat.*, **5(2)**, (2017), 416-427.

[16] **K. Ravi, J.M. Rassias and B.V. Senthil Kumar**, Ulam-Hyers stability of undecic functional equation in quasi-*beta*-normed spaces fixed point method, *Tbilisi Mathematical Science*, **9 (2)**, (2016), 83-103.

[17] **K. Ravi, J.M. Rassias, S. Pinelas and S.Suresh**, General solution and stability of Quattuordecic functional equation in quasi-*beta*-normed spaces, *Advances in pure mathematics*, **6** (2016), 921-941.

[18] **M. Obloza**, Hyers stability of the linear differential equation, *Rockniz Nauk-Dydakt. Prace Mat.*, **13**, (1993), 259-270.

[19] **M. Obloza,** Connections between Hyers and Lyapunov stability of the ordinary differential equations, *Rocznik Nauk-Dydakt, Prace Mat.*, **14**, (1997), 141-146.

[20] **C. Alsina, and R. Ger,** On some inequalities and stability results related to the exponential function, *J. Inequal. Appl.*, **2**, (1998), 373-380.

[21] **S. Takahasi, T. Miura, and S. Miyajima,** On the Hyers-Ulam stability of the Banach space valued differential equation $y'(t) = \lambda\, y(t)$, *Bull. Korean Math. Soc.*, **39**, (2002), 309-315.

[22] **Ginkyu Choi and S.M. Jung,** Invariance of Hyers-Ulam stability of nth order linear differential equations and its applications, *Advances in Difference equations*, (2015) **2015: 277**, 1-14.

[23] **Tongxing Li, Akbar Zada and Shah Faisal**, Hyers-Ulam stability of nth order linear differential equations, *J. Nonlinear Sci. Appl.*, **9** (2016) 2070-2075.

[24] **M.R. Abdollahpour, R. Aghayari and M.Th. Rassias**, Hyers-Ulam stability of associated Laguerre differential Equations in a subclass of analytic functions, *J. Math. Anal. Appl.*, **437** (2016) 605-612.

[25] **S.M. Jung, Jaiok Roh and Juri Lee,** Optimal Hyers-Ulam's Constant for the linear differential equations, *J. Inequal. and Appl.*, (2016) **2016: 201,** 1-7.

[26] **Masakazu Onitsuka and Tomohiro Shoji,** Hyers-Ulam stability of first order homogeneous linear differential equations with a real-valued co-efficient, *Appl. Math. Lett.*, **63** (2017) 102–108.

[27] **R. Murali and A. Ponmana Selvan,** Ulam stability of a LCR Electric circuit with Electromotive force, *Int. J. Math. And Appl.*, **6 (1)** (Special Issue) (2018) 62-66.

[28] **R. Murali and A. Ponmana Selvan,** Hyers-Ulam Stability of Third order System of differential equation, *International Journal of Scientific Research in Mathematical and Statistical Sciences*, **6 (1)**, (2019) 198-202.

[29] **R. Murali and A. Ponmana Selvan,** *Hyers-Ulam-Rassias Stability for the Linear Ordinary Differential Equation of Third order*, Kragujevac J. Math., **42** (4) (2018) 579–590.

[30] **R. Murali and A. Ponmana Selvan,** *Ulam Stability of Third order Linear Differential equations*, International Journal of Pure and Applied Mathematics, **120**(9) (2018), 217–225.

[31] **Murali Ramdoss, Abasalt Bodaghi and Ponmana Selvan Arumugam,** Stability for the third order linear ordinary differential equation, Int. J. Math. Comp., **30, (1)** (2019) 87 - 92.

[32] **R. Murali and A. Ponmana Selvan,** *Hyers-Ulam Stability of nth order linear differential equation*, Proyecciones: Journal of Mathematics (Revista de Matematica) (Antofagasta online), **38**(3) (2019), 553–566.

[33] **R. Murali and A. Ponmana Selvan,** *Hyers-Ulam Stability of a Free and Forced Vibrations*, Kragujevac Journal of Mathematics, **44**(2) (2020), 299–312.

[34] **Murali Ramdoss, Ponmana Selvan-Arumugam and Choonkil Park,** Ulam stability of linear differential equations using Fourier transform, AIMS Mathematics, **5 (2)** (2019) 766-780.

[35] **R. Murali and A. Ponmana Selvan,** *Fourier Transforms and Ulam Stabilities of Linear Differential Equations*, Frontiers in Functional Equations and Analytic Inequalities, Springer Nature, Switzerland, AG 2019, 195–217, 2019.

[36] **J.M. Rassias, R. Murali and A. Ponmana Selvan,** *Mittag-Leffler-Hyers-Ulam Stability of Linear Differential Equations using Fourier Transforms*, J. Computational Analysis and Applications, **29 (1)** (2021) 68-85.