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SOLUTION AND STABILITY OF A RADICAL QUADRATIC FUNCTIONAL EQUATION

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Abstract. In this article, we introduce the radical quadratic functional equation. Also, we obtain their general solution and investigate the generalized Hyers-Ulam-Rassias stability in Modular spaces using fixed point concept.

1. Introduction

Ulam [12] raised the infamous stability problem of functional equations in 1940 at the University of Wisconsin. The solution for the Ulam problem garnered world wide attention and finally came to be identified as generalized Hyers-Ulam, generalized Hyers-Ulam-Rassias, Ulam-Găvruta-Rassias and JMR stabilities of functional equations. One can refer ([2], [4], [6], [10], [9], [11]).

In the probabilistic normed spaces, Mohammad Bagher Ghaemi et al. [5] analyzed the stability for the sextic and quintic mappings.

In the quasi-β-normed spaces via fixed point method, Tian Zhou Xu et al. [14] introduced the following functional equation of quintic type

$$\begin{aligned} g(m+3n) - 5g(m+2n) + 10g(m+n) - 10g(m) \\ + 5g(m-n) - g(m-2n) = 120g(n) \end{aligned}$$

and sextic type

$$\begin{aligned} g(m+3n) - 6g(m+2n) + 15g(m+n) - 20g(m) + 15g(m-n) \\ - 6g(m-2n) + g(m-3n) = 720g(n) \end{aligned}$$

and also investigated their stabilities related to Ulam problem.

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In Felbin spaces, Pasupathi Narasimman et al. [8] introduced generalized sextic and quintic functional equations

$$\begin{aligned} & g(am+n)+g(am-n)+g(m+an)+g(m-an) \\ & \quad =(a^4+a^2)[g(m+n)+g(m-n)]+2(a^6-a^4-a^2+1)[g(m)+g(n)] \end{aligned}$$

$$\begin{aligned} & a[g(am+n)+g(am-n)]+g(m+an)+g(m-an) \\ & \quad =(a^4+a^2)[g(m+n)+g(m-n)]+2(a^6-a^4-a^2+1)g(m) \end{aligned}$$

with general solution and stability for $a \in \mathbb{R}-\{0,\pm 1\}$.

Using fixed point theory, Zamani Eskandani and John Michael Rassias [15], Kittipong Wongkum [13] are obtained modular stability of $\gamma-$quartic and cubic functional equations.

In quasi-β-normed spaces, In Goo Cho et al. [3] analyzed the Ulam stability problem for the quintic functional equation of the form

$$\begin{aligned} & 2g(2m+n)+2g(2m-n)+g(m+2n)+g(m-2n) \\ & \quad =20[g(m+n)+g(m-n)]+90g(m). \end{aligned}$$

In 2015, Abasalt Bodaghi et al.[1] analyzed the general solution and stability of a mixed type of quintic-additive functional equation of the form

$$\begin{aligned} & g(3m+n)-5g(2m+n)+g(2m-n)+10g(m+n)-5g(m-n) \\ & \quad =10g(n)+4g(2m)-8g(m) \end{aligned}$$

in real numbers.

Motivated from the above investigations on functional equations, in this paper we introduce the following new radical quadratic functional equation

$$\begin{aligned} & f(2x+y)+f(2x-y) \\ & \quad =8f(x+y)+8f(x-y)+f(\sqrt{x^2+y^2})-15f(\sqrt{y^2})-9f(\sqrt{x^2}). \end{aligned} \tag{1}$$

Mainly we obtain its general solution and investigate stabilities related to Ulam problem in modular spaces. The definitions related to modular space and fixed point theory to establish our main theorem can be referred in [7].

The paper structured as follows: In Section-2, we obtain the general solution of the functional equation (1). In Section-3, authors discuss generalized Hyers-Ulam-Rassias, Hyers-Ulam and Hyers-Ulam-Rassias stabilities of quadratic functional equation in Modular spaces using fixed point theory. Finally the conclusion given in section-4.

2. General solution of (1)

Theorem 1. *Let X and Y be real vector spaces. If a function $f: X \to Y$ satisfies the functional equation (1) for all $x, y \in X$, then $f: X \to Y$ is even and quadratic.*

Suppose a function $f: X \to Y$ satisfies (1). Putting $x=y=0$ in (1), we get $f(0)=0$. Let $y=0$ in (1), we obtain

$$f(2x)=4f(x) \tag{1}$$

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for all $x \in X$. Let $x = 0$ in (1), we obtain

$$f(-y) = f(y) \tag{2}$$

for all $y \in X$. Hence, $f : X \to Y$ is even. Setting $(x, y) = (x, x)$ and using (1), we obtain

$$f(3x) = 9f(x) \tag{3}$$

for all $x \in X$. From (1) and (3), we arrive

$$f(nx) = n^2 f(x) \tag{4}$$

for all $x \in X$. Hence, $f : X \to Y$ is quadratic.

3. Stability of Functional Equation(1)

In this section, we determine the generalized Hyers-Ulam stability concerning the radical quadratic functional equation (1) in Modular Spaces by using fixed point theory.

For mapping $\rho : M \to X_\xi$, consider

$$\begin{aligned} D_q f(x, y) &:= f(2x + y) + f(2x - y) \\ &\quad - 8f(x + y) - 8f(x - y) - f(\sqrt{x^2 + y^2}) + 15f(\sqrt{y^2}) + 9f(\sqrt{x^2}) \end{aligned}$$

for all $x, y \in M$.

Theorem 2. *Consider a mapping $\rho : M^2 \to [0, +\infty)$ such that*

$$\lim_{n\to\infty} \frac{1}{2^{2n}} 2\rho\{2^n x, 2^n y\} = 0, \tag{1}$$

and

$$\rho\{2x, 2y\} \leq 2^2 \psi \rho\{x, y\}, \forall x, y \in M, \tag{2}$$

for $\psi < 1$. If $f : M \to X_\xi$ fulfill the inequality

$$\xi \left(D_q f(x, y)\right) \leq \rho(x, y), \tag{3}$$

$\forall x, y \in M$. Then $Q : M \to X_\xi$ a unique quadratic mapping exists, such that

$$\xi(Q(x) - f(x)) \leq \frac{1}{2^3(1 - \psi)} \rho(x, 0),\ \forall x \in M. \tag{4}$$

Where M is linear space and X_ξ is modular space which is complete with Fatou property.

Proof. Consider $N = \xi'$ and define ξ' on N as,

$$\xi'(q) =: \inf\{2 > 0 : \xi(f(y)) \leq 2\rho(x, 0), \forall x \in M\}.$$

One can easily prove ξ' is convex modular with Fatou property on N and $N_{\xi'}$ is $\xi-$complete, see [15]. Consider the function $\sigma : N_{\xi'} \to N_{\xi'}$ defined by

$$\sigma f(x) = \frac{1}{2^2} f(2x), \tag{5}$$

for all $x \in M$. Let $p, r \in N_{\xi'}$ and $a \in [0, 1]$ with $\xi'(p - r) < a$. By definition of ξ', we get

$$\xi(p(x) - r(x)) \leq 2\rho(x, 0) \tag{6}$$

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for all $x \in M$. By (2) and (6), we obtain

$$\xi\left(\frac{p(2x)}{2^2} - \frac{r(2x)}{2^2}\right) \leq \frac{1}{2^2}\xi(p(2x) - r(2x)) \leq \frac{1}{2^2}2\rho(2x, 0) \leq 2\psi\rho(x, 0),$$

for all $x \in M$. Hence, σ is a $\xi'-$contraction. From (3), we obtain

$$\xi\left(\frac{f(2x)}{2^2} - f(x)\right) \leq \frac{1}{2^3}\rho(x, 0), \tag{7}$$

for all $x \in M$. Substituting x by $2x$ in (7), we get

$$\xi\left(\frac{f(2^2x)}{2^2} - f(2x)\right) \leq \frac{\rho(2x, 0)}{2^3}, \ \forall x \in M. \tag{8}$$

We obtain from (7) and (8) that

$$\begin{aligned} &\xi\left(\frac{f(2^2x)}{2^4} - f(x)\right) \\ &\leq \xi\left(\frac{f(2^2x)}{2^4} - \frac{f(2x)}{2^2}\right) + \xi\left(\frac{f(2x)}{2^2} - f(x)\right) \\ &\leq \frac{1}{2^5}\rho(2x, 0) + \frac{1}{2^3}\rho(x, 0), \ \forall x \in M. \end{aligned} \tag{9}$$

We get by induction,

$$\begin{aligned} \xi\left(\frac{f(2^nx)}{2^{2n}} - f(x)\right) &\leq \sum_{i=1}^{n} \frac{1}{2 \cdot 2^{2i}}\rho(2^{i-1}x, 0) \\ &\leq \frac{1}{\psi 2^3}\rho(x, 0)\sum_{i=1}^{n}\psi^i \\ &\leq \frac{1}{2^3(1-\psi)}\rho(x, 0), \ \forall x \in M. \end{aligned} \tag{10}$$

We obtain from (10),

$$\begin{aligned} \xi\left(\frac{f(2^nx)}{2^{2n}} - \frac{f(2^sx)}{2^{2s}}\right) &\leq \frac{1}{2}\xi\left(2\frac{f(2^nx)}{2^{2n}} - 2f(x)\right) + \frac{1}{2}\xi\left(2\frac{f(2^sx)}{2^{2s}} - 2f(x)\right) \\ &\leq \frac{\kappa}{2}\xi\left(\frac{f(2^nx)}{2^{2n}} - f(x)\right) + \frac{\kappa}{2}\xi\left(\frac{f(2^sx)}{2^{2s}} - f(x)\right) \\ &\leq \frac{\kappa}{2^3(1-\psi)}\rho(x, 0), \ \forall x \in M \end{aligned} \tag{11}$$

where $n, s \in \mathfrak{N}$. Thus

$$\xi'(\sigma^n f - \sigma^s f) \leq \frac{\kappa}{2^3(1-\psi)},$$

hence the boundedness exists of an orbit of σ at f. $\{\tau^n f\}$ is $\xi'-$converges to $Q \in N_{\xi'}$ by Theorem 1.5 in [15]. By $\xi'-$contractivity of σ, we get

$$\xi'(\sigma^n f - \sigma Q) \leq \psi\xi'(\sigma^{n-1} f - Q).$$

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Allowing $n \to \infty$ and by Fatou property of ξ', we get

$$\begin{aligned}\xi'(\sigma Q - Q) &\leq \lim_{n\to\infty} \inf \xi'(\sigma Q - \sigma^n f)\\ &\leq \psi \lim_{n\to\infty} \inf \xi'(Q - \sigma^{n-1} f) = 0.\end{aligned}$$

Hence, Q is a fixed point of σ. In (3), changing (x, y) by $(2^n x, 2^n y)$, we obtain

$$\xi\left(\frac{1}{2^{2n}} Df(2^n x, 2^n y)\right) \leq \frac{1}{2^{2n}} \rho(2^n x, 2^n y),\ \forall x, y \in M. \tag{12}$$

By Theorem 1 and allowing $n \to \infty$, Q is quadratic and using (10), we arrive (4). For the uniqueness of Q, consider another quadratic mapping $D : M \to X_\xi$ satisfying (4). So that, Q is fixed point of σ.

$$\xi'(Q - D) = \xi'(\sigma Q - \sigma D) \leq \psi \xi'(Q - D). \tag{13}$$

From (13), we get $Q = D$. Hence the proof. □

Proof of following Corollaries 1 and 2 follows that, all normed space implies modular space of modular $\xi(x) = \|x\|$.

Corollary 1. *Assume ρ is a mapping from M^2 to $[0, +\infty)$ for*

$$\lim_{n\to\infty} \frac{1}{2^{2n}} \rho\{2^n x, 2^n y\} = 0, \tag{14}$$

and

$$\rho\{2x, 2y\} \leq 2^2 \psi \rho\{x, y\},\ \forall x, y \in M,\ \psi < 1. \tag{15}$$

If $f : M \to X$ satisfies the condition for X is Banach space

$$\|D_q f(x, y)\| \leq \rho(x, y), \tag{16}$$

$\forall x, y \in M$. Then a unique $Q : M \to X$ quadratic mapping exists, hence

$$\|Q(x) - f(x)\| \leq \frac{\rho(x, 0)}{2^3(1 - \psi)}, \tag{17}$$

for all $x \in M$.

Theorem 3. *Assume that,*

$$\lim_{n\to\infty} \kappa^{2n} \rho\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0, \tag{18}$$

where ρ is a mapping from M^2 to $[0, +\infty)$ and

$$\rho\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{\psi}{2^3} \rho\{x, y\},\ \forall x, y \in M, \psi < 1. \tag{19}$$

If $f : M \to X_\xi$ fulfills the inequality

$$\xi\left(D_q f(x, y)\right) \leq \rho(x, y), \tag{20}$$

$\forall x, y \in M$. Then a unique $Q : M \to X_\xi$ quadratic mapping exists, such that

$$\xi(Q(x) - f(x)) \leq \frac{\psi}{2^3(1 - \psi)} \rho(x, 0),\ \forall x \in M. \tag{21}$$

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Proof. Considering x by $\frac{x}{2}$ in (5) of Theorem 2 and proceeding similar to that of Theorem 2, we complete the proof. □

Corollary 2. *Assume that,*

$$\lim_{n\to\infty} \sigma^{2n} \rho\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0, \tag{22}$$

where ρ is a mapping from M^2 to $[0, +\infty)$ and

$$\rho\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{\psi}{2^2} \rho\{x, y\},\ \forall x, y \in M, \psi < 1. \tag{23}$$

If $f : M \to X$ fulfills the inequality

$$\|D_q f(x, y)\| \leq \rho(x, y), \tag{24}$$

$\forall x, y \in M$. *Then a unique $Q : M \to X$ quadratic mapping exists, such that*

$$\|Q(x) - f(x)\| \leq \frac{\psi}{2^3(1-\psi)} \rho(x, 0),\ \forall x \in M. \tag{25}$$

Using Corollaries 1 and 2, the Hyers-Ulam and generalized Hyers-Ulam stabilities of (1) are obtain in the following corollaries.

Corollary 3. *Assume ρ is a mapping from M^2 to $[0, +\infty)$, X be a Banach space and $\epsilon \geq 0$ be a real number such that*

$$\lim_{n\to\infty} \frac{1}{2^{2n}} \rho\{2^n x, 2^n y\} = 0, \tag{26}$$

and

$$\rho\{2x, 2y\} \leq 2^2 \psi \rho\{x, y\},\ \forall x, y \in M, \psi < 1. \tag{27}$$

If $f : M \to X$ fulfills

$$\|D_q f(x, y)\| \leq \epsilon, \tag{28}$$

$\forall x, y \in M$. *Then a unique $Q : M \to X$ quadratic mapping exists and defined by $Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{2n}}$ so that*

$$\|Q(x) - f(x)\| \leq \frac{\epsilon}{2(2^2 - 1)}, \tag{29}$$

for all $x \in M$ and $a \neq 0, \pm 1$.

Corollary 4. *If $f : M \to X$ fulfills the inequality for M and X are linear space and Banach space, respectively.*

$$\|D_q f(x, y)\| \leq \epsilon \left(\|x\|^p + \|y\|^q\right), \tag{30}$$

$\forall x, y \in M$ *with $0 \leq p, q < 2$ or $p, q > 2$ for some $\epsilon \geq 0$. Then a unique quadratic mapping $Q : M \to X$ exists and defined by $Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{2n}}$, so that*

$$\|Q(x) - f(x)\| \leq \frac{\epsilon}{|2(2^2 - 2^p)|} \|x\|^p,\ \forall x \in M. \tag{31}$$

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4. Conclusion

Mainly, we introduced new radical quadratic functional equation with its general solution and investigated generalized Hyers-Ulam stability in Modular Spaces by using fixed point theory.

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